

A global existence result for a zero Mach number system

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Abstract. This paper is to study global-in-time existence of weak solutions to zero Mach number system which derives from the full Navier-Stokes system, under a special relationship between the viscosity coefficient and the heat conductivity coefficient such that, roughly speaking, the source term in the equation for the newly introduced divergence-free velocity vector field vanishes. In dimension two, thanks to a local-in-time existence result of a unique strong solution in critical Besov spaces given in [20], for arbitrary large initial data, we will show that this unique strong solution exists globally in time, by a weak-strong uniqueness argument.

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1 Introduction

1.1 Derivation of a zero Mach number system

The free evolution of a viscous and heat conducting compressible Newtonian fluid obeys the following Navier-Stokes system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) &= 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \operatorname{div} \sigma + \nabla p &= 0, \\ \partial_t(\rho e) + \operatorname{div}(\rho v e) - \operatorname{div}(k \nabla \vartheta) + p \operatorname{div} v &= \sigma \cdot Dv. \end{cases} \quad (1.1)$$

The above System (1.1) describes the evolution of the mass density $\rho = \rho(t, x)$, the momentum $\rho v = \rho(t, x)v(t, x)$ and the internal energy $\rho e = \rho(t, x)e(t, x)$ respectively, where $t \in \mathbb{R}^+$ is the positive time variable and the space variable x belongs to \mathbb{R}^d with $d \geq 2$. The scalar functions $p = p(t, x)$ and $\vartheta = \vartheta(t, x)$ denote the pressure and the temperature respectively, and the viscous strain tensor σ is given by

$$\sigma := 2\mu S v + \nu \operatorname{div} v \operatorname{Id},$$

where $\mu = \mu(\rho, \vartheta)$ and $\nu = \nu(\rho, \vartheta)$ are the regular viscosity coefficients, Id is the identity tensor and the symmetric rate-of-deformation tensor is denoted by

$$S v := \frac{1}{2}(\nabla v + Dv) \quad \text{with} \quad (\nabla v)_{ij} := \partial_i v^j, \quad (Dv)_{ij} := \partial_j v^i.$$

The thermal conductivity coefficient $k = k(\rho, \vartheta)$ is also a smooth function of ρ and ϑ .

There should be another two equations of state such that System (1.1) is complete and in this work, we will take *ideal gas* model:

$$p = R\rho\vartheta, \quad e = C_v\vartheta, \quad (1.2)$$

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where the two constants R and C_v , denote the ideal gas constant and the specific heat capacity at constant volume respectively.

In this paper we will consider a zero Mach number system, which derives from System (1.1) by letting Mach number vanish and admitting a specific relationship between variable physical coefficients k and μ (see (1.6) below). That is, we assume the fluid to be highly subsonic and hence the compression due to pressure is neglectable and in addition, we restrict ourselves to effectively heat-conducting and viscous fluids.

More precisely, firstly, we introduce the dimensionless Mach number ε as the ratio of the velocity v over the reference sound speed, then the rescaled triplet

$$\left(\rho_\varepsilon(t, x) = \rho\left(\frac{t}{\varepsilon}, x\right), \quad v_\varepsilon(t, x) = \frac{1}{\varepsilon}v\left(\frac{t}{\varepsilon}, x\right), \quad \vartheta_\varepsilon(t, x) = \vartheta\left(\frac{t}{\varepsilon}, x\right) \right)$$

satisfies

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon v_\varepsilon) &= 0, \\ \partial_t(\rho_\varepsilon v_\varepsilon) + \operatorname{div}(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) - \operatorname{div} \sigma_\varepsilon + \frac{\nabla p_\varepsilon}{\varepsilon^2} &= 0, \\ \frac{1}{\gamma-1}(\partial_t p_\varepsilon + \operatorname{div}(p_\varepsilon v_\varepsilon)) - \operatorname{div}(k_\varepsilon \nabla \vartheta_\varepsilon) + p_\varepsilon \operatorname{div} v_\varepsilon &= \varepsilon^2 \sigma_\varepsilon \cdot Dv_\varepsilon, \end{cases} \quad (1.3)$$

where $\gamma := C_p/C_v = 1 + R/C_v$ with the constant C_p denoting the specific heat capacity at constant pressure and

$$\sigma_\varepsilon = 2\mu_\varepsilon S v_\varepsilon + \nu_\varepsilon \operatorname{div} v_\varepsilon \operatorname{Id}, \quad p_\varepsilon = R\rho_\varepsilon \vartheta_\varepsilon, \quad \varsigma_\varepsilon(t, x) = \frac{1}{\varepsilon} \varsigma\left(\frac{t}{\varepsilon}, x\right) \text{ with } \varsigma = \mu, \nu, k \text{ respectively.}$$

Let ε go to 0, that is, the pressure p_ε equals to a positive constant P_0 by Equations (1.3)₂ and (1.3)₃, and hence the rescaled system (1.3) becomes the following zero Mach number system immediately (see [2], [36] or [45] for detailed computation):

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) &= 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \operatorname{div} \sigma + \nabla \Pi &= 0, \\ \operatorname{div} v - \operatorname{div}(\alpha k \nabla \vartheta) &= 0, \end{cases} \quad (1.4)$$

with $\alpha := \frac{\gamma-1}{\gamma P_0}$ being a positive constant. Let us point out here that a rigorous justification of the above low-Mach number limit System (1.4) has been presented by T. Alazard, see [2].

The unknowns in the above system (1.4) turn to be the density ρ , the velocity field v and some unknown “pressure” Π . Thanks to equations of state (1.2), the temperature ϑ is thus equal to $P_0/(R\rho)$. Consequently, all the physical coefficients k, μ, ν can be viewed as regular functions of the density ρ only. Set $\kappa = \kappa(\rho) = \alpha k \vartheta$, then Equation (1.4)₃ becomes

$$\operatorname{div}(v + \kappa \nabla \ln \rho) = 0. \quad (1.5)$$

In order to take advantage of the “almost divergence-free” condition of Equation (1.4)₃, just as in [20], we introduce a new solenoidal “velocity” field u as follows:

$$u = v - \alpha k \nabla \vartheta = v + \kappa \nabla \ln \rho.$$

We next try to rewrite the terms concerning the original velocity field v in System (1.4) in light of the newly introduced velocity u . Firstly, it is easy to see that

$$\rho v = \rho u - \kappa \nabla \rho.$$

It is hence easy to write $\partial_t(\rho v)$ as (noticing that κ can be viewed as a regular function of ρ only):

$$\partial_t(\rho v) = \partial_t(\rho u) - \nabla(\kappa \partial_t \rho).$$

We can also rewrite the convection term in the conservation law of momentum as following:

$$\operatorname{div}(\rho v \otimes v) = \operatorname{div}(\rho v \otimes u) + \operatorname{div}(v \otimes (-\kappa \nabla \rho)).$$

Observing the following two equalities

$$Sv - Dv = Au := \frac{1}{2}(\nabla u - Du) \quad \text{and} \quad -\operatorname{div}(\nu \operatorname{div} v \operatorname{Id}) = -\nabla(\nu \operatorname{div} v),$$

the viscosity term $-\operatorname{div} \sigma$ thus reads as

$$-\operatorname{div}(2\mu Au) - \operatorname{div}(2\mu Dv) - \nabla(\nu \operatorname{div} v) = -\operatorname{div}(2\mu Au) + \operatorname{div}(v \otimes 2\nabla \mu) - \nabla(\operatorname{div}(2\mu v) + \nu \operatorname{div} v).$$

Therefore, if we admit the following relationship between the heat conductivity coefficient k and the viscosity coefficient μ :

$$-\kappa(\rho) + 2\mu'(\rho) = 0, \quad \text{i.e.} \quad k(\vartheta) + 2C_p \vartheta \mu'(\vartheta) = 0, \quad (1.6)$$

then by introducing a new “pressure” π as

$$\pi = \Pi - \kappa \partial_t \rho - \operatorname{div}(2\mu v) - \nu \operatorname{div} v,$$

System (1.4) recasts in the following system for the new unknown triplet $(\rho, u, \nabla \pi)$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = \partial_t \rho + \operatorname{div}(\rho u) - \operatorname{div}(\kappa \nabla \rho) & = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho v \otimes u) - \operatorname{div}(2\mu Au) + \nabla \pi & = 0, \\ \operatorname{div} u & = 0. \end{cases} \quad (1.7)$$

It is easy to check that if the triplet $(\rho, u, \nabla \pi)$ solves the above system (1.7) in distribution sense which, also satisfies

$$\begin{aligned} \rho(t, x) &\in [\underline{\rho}, \overline{\rho}], \quad \forall t \in [0, +\infty), \quad x \in \mathbb{R}^d, \\ \rho - c_0 &\in L^\infty([0, +\infty); L^2(\mathbb{R}^d)) \cap L^2_{\text{loc}}([0, +\infty); H^1(\mathbb{R}^d)), \\ \partial_t \rho &\in L^2_{\text{loc}}([0, +\infty); H^{-1}(\mathbb{R}^d)), \\ u, v := u - \kappa \nabla \ln \rho &\in L^2_{\text{loc}}([0, +\infty); (L^2(\mathbb{R}^d))^d), \end{aligned} \quad (1.8)$$

with three positive constants $\underline{\rho}, \overline{\rho}, c_0$, then the above calculation from System (1.4) to System (1.7) makes sense and (ρ, v) , together with some distribution pressure Π , satisfies System (1.4) at least in the sense of distribution, under the coefficient condition (1.6).

Notice that System (1.4) also can be viewed as Kazhikhov-Smagulov type models, which describe the motion of mixtures such as a salt or pollutant spread in a compressible fluid, see [5]. There, the divergence of the velocity v is related to the derivatives of the density ρ by Fick's law

$$\operatorname{div} v + \operatorname{div}(\kappa_0 \nabla \ln \rho) = 0, \quad (1.9)$$

where the diffusive coefficient κ_0 is a positive constant. Therefore, in that case, by view of (1.5), Relation (1.6) implies that the viscosity coefficient μ equals to $\kappa_0 \rho / 2$, up to a constant. Another interesting example proposed in [39] is a low Mach number combustion model with a constant thermic coefficient k , where Relation (1.6) entails $\mu = -k \ln \vartheta / (2C_p)$, up to a constant. More generally, by virtue of (1.6), we consider the gases which become less viscous as the temperature increases (or the density decreases) and moreover, if there is more thermal conduction, the viscosity decreases faster. However, unlike the physical coefficients k and μ , the other viscosity coefficient ν plays no role here.

1.2 Main results

In this work, we want to consider Cauchy problem of System (1.4) globally in time. There is a long history of global existence problem of weak solutions (roughly speaking, solutions in the sense of distribution with bounded physical energy) with *large* initial data to Navier-Stokes system. As early as in 1934, J. Leray in [35] proved such a global existence result to the classical incompressible Navier-Stokes equations (System (1.4) with constant density and temperature) in dimension $d = 2$ and 3, and especially in dimension 2 global regularity and uniqueness also hold. Another big breakthrough is due to P.-L. Lions, who treated a somehow simplified compressible Navier-Stokes system (the first two equations in System (1.1) under the gamma-type pressure law $p(\rho) = a\rho^\gamma$, $a > 0$) in the nineties of last century. Indeed in [37], observing the so-called effective viscous flux, he showed that if $\gamma \geq 9/5$ in dimension 3, weak solutions exist globally in time. In the very beginning of this century, E. Feireisl improved this result to the case $\gamma > 3/2$, see [26]. However, for the compressible case, global regularity or uniqueness for large data is still unknown, even in dimension $d = 2$, due to a lack of estimation in the vacuum region. Recently in [10], D. Bresch and B. Desjardins extended this type of result to the full Navier-Stokes system (1.1) by using the so-called BD-entropy, under some specific assumptions on equations of state and physical coefficients which are different from the coefficient relationship (1.6) here. Let us emphasize also that, in [27], E. Feireisl presented a global existence result for System (1.1), but referring to the so-called “variational” solutions.

On the other side, there are also numerous works involving global-in-time existence of strong solutions (unique and regular in general) to Navier-Stokes system, but with *small* initial data in general. Let’s just mention that for the classical incompressible Navier-Stokes equations, in 1964, H. Fujita and T. Kato obtained a unique global solution, see [29]. With high regularity assumptions, in dimension 3, A. Matsumura and T. Nishida [40] studied the motion of viscous and heat-conductive gases. In [14] and [15], R. Danchin considered the movement of barotropic compressible fluids and compressible viscous and heat-conducting gases respectively, but in critical Besov spaces.

When there is no heat conduction, System (1.4) becomes the density-dependent incompressible Navier-Stokes system. See the book [3] and the references therein for the existence results of the associated initial-boundary value problem. Global well-posedness of the Cauchy problem was demonstrated by R. Danchin in [17] and in addition in dimension 2, he also examined *large* initial data case in [18]. There are also a few works devoted to global-in-time solutions to the Cauchy problem of zero Mach number system (1.4), under some smallness assumptions on the diffusion coefficient κ or on the initial data in general. In the pioneering work [33], A.V. Kazhikhov and Sh. Smagulov addressed the initial boundary value problem of a somehow *simplified* mathematical model for a two-component mixture involving mass diffusion inside. More precisely, the constant diffusion coefficient κ_0 in the Fick law (1.9) was assumed to be *small* compared with the *constant* viscosity coefficient μ . Besides, after taking the transformation (1.1), they just neglected the term $\operatorname{div}(\kappa_0 \nabla \ln \rho \otimes (-\kappa_0 \nabla \rho))$ in the convection term in Equation (1.4)₂, which is of order κ_0^2 . Finally, for the system (1.7) with an additional convection term $-\kappa_0 u \cdot \nabla^2 \rho$ on the left side, they obtained a global-in-time generalized solution for finite-energy initial data with the initial inhomogeneity $\rho_0 - 1$ in H^1 . If in addition the initial velocity u_0 belongs to H^1 , this global solution is unique in dimension two. Under a similar smallness hypothesis on κ_0 , P. Secchi [43] got a unique global classical solution in dimension two for System (1.4). There, he also studied the asymptotic behaviour when κ_0 goes to zero. The general case (i.e. κ_0 can be variable and arbitrarily large) was considered by Beirão da Veiga [5, 7]. See also [9] for the inviscid case. Under exactly the condition (1.6) on the physical coefficients, [11, 12] gave the existence of global-in-time weak solutions in smooth bounded domains. Furthermore in dimension two, a recent work [13] proved that the weak solutions are in fact unique, under the same initial condition as in Kazhikhov-Smagulov [33] above. Let us also mention here that for a low Mach number combustion model proposed in [39] which, in addition to System (1.4), also takes into account a reaction-diffusion equation for the different

species, P. Embid [25] got a unique *local-in-time* regular solution. In Section 8.8 in [37], P.-L. Lions indicated that in dimension 2, *small* and smooth enough perturbation will indeed evolve a global weak solution to System (1.4), if the heat conductivity coefficient k is a positive constant. Recently in [20], R. Danchin and the author studied System (1.4) where *variable* positive physical coefficients case was considered. There, without assuming any specific coefficient relationship, a global strong solution was obtained, for *small* initial perturbation in any dimension $d \geq 2$.

At last, let us just mention some results on the low Mach number limit. After the pioneer works by D.G. Ebin [23, 24] and S. Klainerman and A. Majda [34], the incompressible limit has attracted many mathematicians' attention. Among them, we cite [1, 31, 32, 41, 42, 44] for the inviscid case while [2, 6, 16, 21, 22, 28, 30, 38] for Navier-Stokes equations. See also [8] for a good summary.

The aim of this paper is to prove the global-in-time existence of weak solutions of Leray's type to zero Mach number system (1.4) where the two positive variable physical coefficients k, μ satisfy Condition (1.6), on the whole space \mathbb{R}^d , $d \geq 2$. Furthermore, thanks to the local-in-time existence of a unique strong solution to System (1.4) given in [20], we can also show that in dimension 2, for any initial data of critical regularity, the weak solutions are in fact strong solutions and hence unique and regular. This is, to our knowledge, the first result of this kind for System (1.4).

Let's first analyze System (1.7) *formally*. If we assume the initial density $\rho_0(x)$ to be bounded from below and above by two positive constants $\underline{\rho}$ and $\bar{\rho}$ respectively, then the maximum principle for the parabolic equation $(1.7)_1$ ensures the density to satisfy the following uniform bound:

$$\rho(t, x) \in [\underline{\rho}, \bar{\rho}], \quad \forall t \geq 0, x \in \mathbb{R}^d. \quad (1.10)$$

Furthermore, if ρ is close to a constant, say "1", at infinity, then we expect the solution (ρ, u) to satisfy the following two energy equalities which come from taking the $L^2(\mathbb{R}^d)$ -inner product between $\rho - 1$ (resp. u) and $(1.7)_1$ (resp. $(1.7)_2$):

$$\int_{\mathbb{R}^d} |\rho(t) - 1|^2 + 2 \int_0^t \int_{\mathbb{R}^d} \kappa |\nabla \rho|^2 = \int_{\mathbb{R}^d} |\rho_0 - 1|^2, \quad \forall t > 0, \quad (1.11)$$

and

$$\int_{\mathbb{R}^d} \rho(t) |u(t)|^2 + 4 \int_0^t \int_{\mathbb{R}^d} \mu |Au|^2 = \int_{\mathbb{R}^d} \rho_0 |u_0|^2, \quad \forall t > 0. \quad (1.12)$$

Therefore, we complement System (1.7) with initial data (ρ_0, u_0) verifying

$$\rho - 1|_{t=0} = \rho_0 - 1 \in L^2(\mathbb{R}^d), \quad u|_{t=0} = u_0 \in (L^2(\mathbb{R}^d))^d, \quad 0 < \underline{\rho} \leq \rho_0 \leq \bar{\rho}, \quad \operatorname{div} u_0 = 0, \quad (1.13)$$

hoping that the obtained solution (ρ, u) satisfies (1.11) and (1.12), at least in inequality form.

Moreover, we assume that under the bounds (1.10) imposed on the density, there exist positive constants $\underline{k}, \bar{k}, \underline{\mu}, \bar{\mu}$ depending only on $\underline{\rho}, \bar{\rho}$ such that the physical coefficients k and μ also have positive lower and upper bounds:

$$0 < \underline{k} \leq k(t, x) \leq \bar{k}, \quad 0 < \underline{\mu} \leq \mu(t, x) \leq \bar{\mu}, \quad \forall t \geq 0, x \in \mathbb{R}^d. \quad (1.14)$$

To conclude, we have the following global existence result for System (1.7):

Theorem 1.1. *There exists a global-in-time weak solution (ρ, u) to Cauchy problem (1.7)-(1.13) in the following sense:*

- $\rho - 1 \in C([0, +\infty); L^p(\mathbb{R}^d)) \cap L^2_{\text{loc}}([0, +\infty); H^1(\mathbb{R}^d)), \forall p \in [2, \infty)$.
 ρ satisfies $(1.7)_1$ in $L^2_{\text{loc}}([0, +\infty); H^{-1}(\mathbb{R}^d))$ and $\rho(0) = \rho_0$.
The uniform bound (1.10) and Energy Equality (1.11) both hold.

- For any $t > 0$ and any test function $\phi(t, x) \in (C^\infty([0, +\infty) \times \mathbb{R}^d))^d$ with compact support such that $\operatorname{div} \phi = 0$, the following holds:

$$\int_{\mathbb{R}^d} \rho(t) u(t) \cdot \phi - \int_{\mathbb{R}^d} \rho_0 u_0 \cdot \phi|_{t=0} - \int_0^t \int_{\mathbb{R}^d} [\rho u \cdot \partial_\tau \phi + (\rho u - \kappa \nabla \rho) \cdot \nabla \phi \cdot u - 2\mu A u : A \phi] = 0. \quad (1.15)$$

- $u \in C([0, \infty); (L_w^2(\mathbb{R}^d))^d) \cap L_{\text{loc}}^2([0, +\infty); (H^1(\mathbb{R}^d))^d)$, $u(0) = u_0$ and $\operatorname{div} u = 0$ in $L_w^2(\mathbb{R}^+ \times \mathbb{R}^d)$ with L_w^2 denoting the Lebesgue space L^2 endowed with weak topology. There exists a positive constant C depending only on $\underline{\rho}, \bar{\rho}$ such that u verifies the energy inequality

$$\int_{\mathbb{R}^d} |u(t)|^2 + \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 \leq C \int_{\mathbb{R}^d} |u_0|^2, \quad \forall t > 0. \quad (1.16)$$

In the above, it is easy to check (by view of Condition (1.8)) that ρ and $v := u - \kappa \nabla \ln \rho \in L^2([0, +\infty); (L^2(\mathbb{R}^d))^d)$ satisfy System (1.4)-(1.6) in distribution sense. But owing to a lack of high regularity assumption on the initial density ρ_0 , we don't know the continuity of v at the initial instant, since we can't even define the quantity $\kappa(\rho_0) \nabla \ln \rho_0$.

Instead, if we assume $\rho_0 - 1 \in H^1(\mathbb{R}^d)$ in addition to the initial condition (1.13), that is, the initial original velocity field v_0 belongs to $(L^2(\mathbb{R}^d))^d$ too, then we expect that there exists a global weak solution $(\rho - 1, v) \in H^1(\mathbb{R}^d) \times (L^2(\mathbb{R}^d))^d$ to System (1.4)-(1.6), if $d = 2$ or 3 . In fact, if $\kappa \equiv 1$, then taking $L^2(\mathbb{R}^d)$ -inner product between Equation (1.7)₁ and $\Delta \rho$ yields

$$\frac{1}{2} \frac{d}{dt} \|\nabla \rho\|_{L^2(\mathbb{R}^d)}^2 + \|\Delta \rho\|_{L^2(\mathbb{R}^d)}^2 \leq \left| \int_{\mathbb{R}^d} \operatorname{div}(\rho u) \Delta \rho \right|. \quad (1.17)$$

Since a priori we have the estimate (noticing $\operatorname{div} u = 0$)

$$\left| \int_{\mathbb{R}^d} \operatorname{div}(\rho u) \Delta \rho \right| = \left| \int_{\mathbb{R}^d} \nabla \rho \cdot \nabla u \cdot \nabla \rho \right| \leq \|\nabla u\|_{L^2(\mathbb{R}^d)} \|\nabla \rho\|_{L^4(\mathbb{R}^d)}^2,$$

and the following two interpolation inequalities

$$\|\nabla \rho\|_{L^4(\mathbb{R}^2)}^2 \lesssim \|\Delta \rho\|_{L^2(\mathbb{R}^2)} \|\nabla \rho\|_{L^2(\mathbb{R}^2)}, \quad \|\nabla \rho\|_{L^4(\mathbb{R}^3)}^2 \lesssim \|\Delta \rho\|_{L^2(\mathbb{R}^3)} \|\rho\|_{L^\infty(\mathbb{R}^3)}, \quad (1.18)$$

we have from Young's Inequality and Estimate (1.16) the following two energy estimates:

$$\begin{aligned} \|\nabla \rho\|_{L_t^\infty(L^2(\mathbb{R}^2))} + \|\Delta \rho\|_{L_t^2(L^2(\mathbb{R}^2))} &\leq C \|\nabla \rho_0\|_{L^2} e^{C \int_0^t \|\nabla u\|_{L^2}^2} \leq C \|\nabla \rho_0\|_{L^2(\mathbb{R}^2)} e^{C \|u_0\|_{L^2(\mathbb{R}^2)}^2}, \\ \|\nabla \rho\|_{L_t^\infty(L^2(\mathbb{R}^3))} + \|\Delta \rho\|_{L_t^2(L^2(\mathbb{R}^3))} &\leq C (\|\nabla \rho_0\|_{L^2} + \|\nabla u\|_{L_t^2(L^2)}) \leq C \|(\nabla \rho_0, u_0)\|_{L^2(\mathbb{R}^3)}, \end{aligned} \quad (1.19)$$

for some constant C depending also on $\underline{\rho}, \bar{\rho}$. In general case where κ depends on ρ , we consider, instead, the equation for the scalar function $K = K(\rho)$ with $\nabla K = \kappa \nabla \rho$ and $K(1) = 0$, see Section 2.2 in the following for more details.

But it is not clear that we can still have $v \in L_t^\infty((L^2(\mathbb{R}^d))^d)$, in dimension $d \geq 4$, due to a lack of an interpolation inequality like (1.18) which can be used to control the convection term $u \cdot \nabla \rho$ in the equation of the density. Anyway, we have

Theorem 1.2. *Let $d = 2, 3$ and Relation (1.6) hold. For any initial data (ρ_0, v_0) such that*

$$\rho_0 - 1 \in H^1(\mathbb{R}^d), \quad v_0 \in (L^2(\mathbb{R}^d))^d, \quad 0 < \underline{\rho} \leq \rho_0 \leq \bar{\rho}, \quad \operatorname{div}(v_0 + \kappa(\rho_0) \nabla \ln \rho_0) = 0, \quad (1.20)$$

there exists a global-in-time weak solution (ρ, v) to Cauchy problem (1.4)-(1.20) in the sense given in Theorem 1.1, except with Equality (1.15) replaced by

$$\int_{\mathbb{R}^d} \rho(t) v(t) \cdot \phi - \int_{\mathbb{R}^d} \rho_0 v_0 \cdot \phi|_{t=0} - \int_0^t \int_{\mathbb{R}^d} [\rho v \cdot \partial_\tau \phi + \rho v \cdot \nabla \phi \cdot v - \sigma : D \phi] = 0. \quad (1.21)$$

Furthermore, (ρ, v) satisfies

$$\begin{aligned} (\rho - 1, v) &\in C\left([0, +\infty); H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2\right), \\ \rho - 1 &\in C\left([0, +\infty); H^s(\mathbb{R}^3)\right), \forall s < 1, \quad \text{and} \quad v \in C\left([0, +\infty); (L_w^2(\mathbb{R}^3))^3\right). \end{aligned} \quad (1.22)$$

and the following energy estimate:

$$\|(\rho - 1, v)\|_{L_t^\infty([0, \infty); H^1 \times L^2)} + \|(\nabla \rho, \nabla v)\|_{L_t^2([0, \infty); H^1 \times L^2)} \leq C(\underline{\rho}, \bar{\rho}, \|(\rho_0 - 1, v_0)\|_{H^1 \times L^2}). \quad (1.23)$$

Next we want to show the global solutions got above are actually unique for some reasonably smooth initial data when $d = 2$. We notice that even if $\rho - 1 \in L^\infty(H^1)$, it is difficult to show the uniqueness. In fact, if we consider the system for the difference of any two solutions, then the nonlinear terms in System (1.4) ask the $L^\infty(L^\infty)$ -norm control on the difference of two densities. It is unknown a priori because the difference does not (at least not obviously) satisfy any parabolic equation and another unlucky thing is that we can't embed $H^1(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$. Therefore, we have to resort to Besov functional space $B_{2,1}^1(\mathbb{R}^2)$ (see Definition 1.1 below) which can be embedded both in H^1 and in L^∞ in dimension 2 and furthermore, we have already the following existence result, as a special case of Theorem 1.2 in [20]:

Theorem 1.3. *In dimension 2, for any initial density ρ_0 and velocity field u_0 which satisfy*

$$0 < \underline{\rho} \leq \rho_0 \leq \bar{\rho}, \quad \operatorname{div} u_0 = 0 \quad \text{and} \quad \|\rho_0 - 1\|_{B_{2,1}^1} + \|u_0\|_{B_{2,1}^0} \leq M, \quad (1.24)$$

for some positive constants $\underline{\rho}, \bar{\rho}, M$, there exists a positive time T_c depending only on $\underline{\rho}, \bar{\rho}, M$ such that System (1.7) has a unique solution $(\rho, u, \nabla \pi)$ satisfying

$$\rho \in [\underline{\rho}, \bar{\rho}], \quad \|(\rho - 1, u, \nabla \pi)\|_{E_{T_c}} \leq CM, \quad (1.25)$$

where C is some positive constant and the solution space E_T is the following critical nonhomogeneous Besov spaces

$$E_T \triangleq \left(C([0, T]; B_{2,1}^1) \cap L_T^1(B_{2,1}^3)\right) \times \left(C([0, T]; B_{2,1}^0) \cap L_T^1(B_{2,1}^2)\right)^2 \times \left(L_T^1(B_{2,1}^0)\right)^2.$$

Hence, System (1.7)-(1.24) admits a unique local strong solution $(\rho, u, \nabla \pi)$ on its lifespan $[0, T^*)$, $T^* > T_c$, with $(\rho - 1, u, \nabla \pi) \in E_t$ for any $t < T^*$. Moreover, there exists a positive time $T_0 < T^*$ such that $(\rho - 1, u)|_{t=T_0} \in B_{2,1}^3 \times (B_{2,1}^2)^2 \subset H^2 \times (H^1)^2$. Just as in [18], we therefore consider an extra pseudo-conservation law concerning $L^\infty(H^2) \times (L^\infty(H^1))^2$ -norm of the weak solutions which evolve from the initial moment T_0 . This law ensures that the global weak solutions belong to the (strong) solution space E_t for all $t \geq T_0$, by virtue of the embedding $H^2 \times (H^1)^2 \subset B_{2,1}^1 \times (B_{2,1}^0)^2$, see Lemma 3.2 for more details. Thanks to the uniqueness of the strong solutions on the time interval $[0, T^*)$, the Cauchy problem (1.7)-(1.24) has a unique global strong solution $(\rho, u, \nabla \pi)$ with $(\rho - 1, u, \nabla \pi)$ in E_T for all $T \in (0, +\infty)$. By use of Equation (1.7)₁ and the following estimates in Besov spaces (see Section 1.3):

$$\|f(\rho) - f(1)\|_{B_{2,1}^s} \leq C(\underline{\rho}, \bar{\rho})\|\rho - 1\|_{B_{2,1}^s}, \quad \forall f \in C^1, \quad s > 0, \quad \|gh\|_{B_{2,1}^a} \leq C\|g\|_{B_{2,1}^1}\|h\|_{B_{2,1}^a}, \quad \text{if } a = 0, 1,$$

it is easy to find that $\partial_t \rho \in L^1([0, +\infty); B_{2,1}^1(\mathbb{R}^2))$ and furthermore, by view of the calculation from System (1.4) to System (1.7), we have

Theorem 1.4. *Let $d = 2$ and Relation (1.6) hold. For any initial data (ρ_0, v_0) satisfying*

$$0 < \underline{\rho} \leq \rho_0 \leq \bar{\rho}, \quad \operatorname{div}(v_0 + \kappa_0 \nabla \ln \rho_0) = 0, \quad \text{and} \quad \rho_0 - 1 \in B_{2,1}^1(\mathbb{R}^2), \quad v_0 \in (B_{2,1}^0(\mathbb{R}^2))^2, \quad (1.26)$$

System (1.4) has a unique global strong solution $(\rho, v, \nabla \Pi)$ with $(\rho - 1, v, \nabla \Pi)$ in Space E_T for any $T \in (0, +\infty)$.

Remark 1.1. *Let us make some remarks here:*

- If $(\rho, v, \nabla \Pi)$ is a solution of (1.4), then so does

$$(\rho(\ell^2 t, \ell x), \ell v(\ell^2 t, \ell x), \ell^3 \nabla \Pi(\ell^2 t, \ell x)) \quad \text{for all } \ell > 0. \quad (1.27)$$

Therefore, the initial data (ρ_0, v_0) given in (1.26) is of critical regularity in the sense that $\dot{B}_{2,1}^1 \times \dot{B}_{2,1}^0$ -norm is invariant by the transform

$$(\rho_0, v_0)(x) \rightarrow (\rho_0(\ell x), \ell v_0(\ell x)) \quad \text{for all } \ell > 0. \quad (1.28)$$

- Theorem 1.4 implies that in dimension two, for any initial datum $\rho_0 \geq \underline{\rho} > 0$ such that $\rho_0 - 1 \in B_{2,1}^1$, there exists a unique global-in-time solution to the quasilinear heat equation $\partial_t \rho - \operatorname{div}(\kappa(\rho) \nabla \rho) = 0$ in functional space $C_b([0, +\infty); B_{2,1}^1) \cap L^1([0, +\infty); B_{2,1}^3)$.

1.3 Besov spaces and some notations

For completeness, let's define Besov spaces here. Fix a nonincreasing radial function χ in functional space $C_0^\infty(\mathbb{R}^d)$, $d \geq 2$, such that it is supported on the ball $B(0, \frac{4}{3})$ and equals to 1 in a neighborhood of the unit ball $B(0, 1)$. Define a sequence of functions $\{\chi_j\}_{j \geq -1}$ in light of χ as

$$\chi_{-1} = \chi \quad \text{and} \quad \chi_j(\xi) = \chi(\xi/2^{j+1}) - \chi(\xi/2^j) \quad \text{if } j \geq 0.$$

For all $j \geq -1$, denote the inverse Fourier transformation of χ_j by h_j . Therefore we can define the dyadic blocks $(\Delta_j)_{j \geq -1}$ as

$$\Delta_j u = \chi_j(D)u = \int_{\mathbb{R}^d} h_j(y)u(x-y)dy \quad \text{if } j \geq -1.$$

Hence we can now define the Besov space $B_{p,r}^s$:

Definition 1.1. For $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^d)$, we define

$$B_{p,r}^s(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d), \|u\|_{B_{p,r}^s} := \left\| (2^{js} \|\Delta_j u\|_{L^p(\mathbb{R}^d)})_{j \geq -1} \right\|_{\ell^r} < \infty \right\}.$$

We have some basic properties of Besov spaces:

Proposition 1.1. Let $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$ and f be a continuous function from \mathbb{R} to \mathbb{R} such that $f(0) = 0$. Then there exists a positive constant C such that

$$\|\nabla u\|_{B_{p,r}^{s-1}(\mathbb{R}^d)} \leq C \|u\|_{B_{p,r}^s(\mathbb{R}^d)},$$

and (here C depends also on $\|u\|_{L^\infty}$)

$$\|f \circ u\|_{B_{p,r}^s(\mathbb{R}^d)} \leq C \|u\|_{B_{p,r}^s(\mathbb{R}^d)}, \quad \text{if } s > 0. \quad (1.29)$$

Since we will generally work in Sobolev spaces, we state some embedding results (see [4]):

Proposition 1.2. The following imbeddings hold true:

- (i) $B_{p_1, r_1}^{s_1}(\mathbb{R}^d) \hookrightarrow B_{p_2, r_2}^{s_2}(\mathbb{R}^d)$ whenever $s_1 - \frac{d}{p_1} > s_2 - \frac{d}{p_2}$, $p_1 \leq p_2$ or when $s_1 = s_2$, $p_1 = p_2$ and $r_1 \leq r_2$.
- (ii) The classical Sobolev space $H^s(\mathbb{R}^d)$ can be represented by $B_{2,2}^s(\mathbb{R}^d)$ and $H^{s+1}(\mathbb{R}^d) \hookrightarrow B_{2,1}^s(\mathbb{R}^d)$, $B_{2,1}^s(\mathbb{R}^d) \hookrightarrow H^s(\mathbb{R}^d)$.

(iii) In dimension d , spaces of the form $B_{p,1}^{\frac{d}{p}}(\mathbb{R}^d)$ with any $p \in [1, \infty]$ can be imbedded in $L^\infty(\mathbb{R}^d)$.

We recall here an apriori estimate for the following linear parabolic equation in Besov spaces in dimension 2 (see Proposition 4.1 in [20]¹):

Proposition 1.3. *Let $s \in (-1, 1]$. Let $a(t, x) \in \mathcal{S}([0, T] \times \mathbb{R}^2)$ satisfy*

$$\begin{cases} \partial_t a + u \cdot \nabla a - \operatorname{div}(\kappa \nabla a) &= f, \\ a|_{t=0} &= a_0, \end{cases} \quad (1.30)$$

where $u = u(t, x) \in \mathbb{R}^2$, $\kappa = \kappa(t, x) \in \mathbb{R}^+$ are known and $\kappa \geq \underline{\kappa} > 0$. Then there exists a constant C depending on $\underline{\kappa}$ and s such that for any $t \in (0, T]$,

$$\begin{aligned} \|a\|_{L_t^\infty(B_{2,1}^s(\mathbb{R}^2)) \cap L_t^1(B_{2,1}^{s+2}(\mathbb{R}^2))} &\leq \left(\|a_0\|_{B_{2,1}^s(\mathbb{R}^2)} + C \|\Delta_{-1} a\|_{L_t^1(L^2(\mathbb{R}^2))} + \|f\|_{L_t^1(B_{2,1}^s(\mathbb{R}^2))} \right) \\ &\quad \times \exp \left\{ C \left(\|\nabla u\|_{L_t^1(B_{2,1}^1(\mathbb{R}^2))} + \|\nabla \kappa\|_{L_t^2(B_{2,1}^1(\mathbb{R}^2))}^2 \right) \right\}. \end{aligned} \quad (1.31)$$

We introduce also an estimation for products in Besov spaces in dimension 2, which is needed in dealing with nonlinear terms:

Proposition 1.4. *Let $s \in (-1, 1]$. Then there exists some positive constant C such that*

$$\|ab\|_{B_{2,1}^s(\mathbb{R}^2)} \leq C \|a\|_{B_{2,1}^1(\mathbb{R}^2)} \|b\|_{B_{2,1}^s(\mathbb{R}^2)}. \quad (1.32)$$

Let's give the proof for the reader's convenience. Firstly, let's focus on the paraproduct

$$T_a b := \sum_{j \geq -1} \left(\sum_{j' \leq j-2} \Delta_{j'} a \right) \Delta_j b.$$

Since the Fourier transform of $(\sum_{j' \leq j-2} \Delta_{j'} a \Delta_j b)$ is supported near the annulus of size 2^j centered at the origin, it is easy to see that for any $s \in \mathbb{R}$,

$$\|T_a b\|_{B_{2,1}^s} \leq C \sum_j 2^{js} \left\| \sum_{j' \leq j-2} \Delta_{j'} a \right\|_{L^\infty} \|\Delta_j b\|_{L^2} \leq C \|a\|_{L^\infty} \|b\|_{B_{2,1}^s} \leq C \|a\|_{B_{2,1}^1} \|b\|_{B_{2,1}^s}.$$

Similarly, if $s = 1$, then the above also holds for $T_b a$. Otherwise, if $s < 1$, then we can calculate the paraproduct $T_b a$ as following:

$$\begin{aligned} \|T_b a\|_{B_{2,1}^s(\mathbb{R}^2)} &\leq C \sum_j 2^{js} \left(\sum_{j' \leq j-2} \|\Delta_{j'} b\|_{L^\infty} \right) \|\Delta_j a\|_{L^2} \\ &\leq C \sum_j \left(\sum_{j' \leq j-2} 2^{(j-j')(s-1)} 2^{j'(s-1)} \|\Delta_{j'} b\|_{L^\infty} \right) 2^j \|\Delta_j a\|_{L^2} \\ &\leq C \|b\|_{B_{\infty,1}^{s-1}(\mathbb{R}^2)} \|a\|_{B_{2,1}^1(\mathbb{R}^2)} \leq C \|a\|_{B_{2,1}^1(\mathbb{R}^2)} \|b\|_{B_{2,1}^s(\mathbb{R}^2)}. \end{aligned}$$

At last, we consider the remainder

$$R(a, b) := \sum_q \left(\Delta_q a \sum_{q-1 \leq q' \leq q+1} \Delta_{q'} b \right).$$

Since the Fourier transform of $(\Delta_q a \sum_{q-1 \leq q' \leq q+1} \Delta_{q'} b)$ is supported near the ball of size 2^q centered at the origin, one easily finds that if $s > -1$, then

$$\begin{aligned} \|R(a, b)\|_{B_{2,1}^s(\mathbb{R}^2)} &\leq \|R(a, b)\|_{B_{1,1}^{s+1}(\mathbb{R}^2)} \leq C \sum_j 2^{j(s+1)} \sum_{q \geq j-2} \|\Delta_q a\|_{L^2} \sum_{q-1 \leq q' \leq q+1} \|\Delta_{q'} b\|_{L^2} \\ &\leq C \sum_j \sum_{q \geq j-2} 2^{j(s+1)} \left(2^{-q} \|a\|_{B_{2,1}^1} \right) \left(2^{-qs} \|b\|_{B_{2,1}^s} \right) \leq C \|a\|_{B_{2,1}^1} \|b\|_{B_{2,1}^s}. \end{aligned}$$

This completes the proof of Proposition 1.4.

¹The a priori estimates can be extended to any dimension $d \geq 2$ and more general Besov spaces.

Notations

Let us fix some notations which will be used throughout in the sequel:

- We always take $\varrho = \rho - 1$ in any environment, that is, $\varrho_0 = \rho_0 - 1$, $\varrho^\varepsilon = \rho^\varepsilon - 1$, etc.
- We will always view the physical coefficients as functions of the density ρ and denote for example, $\kappa' \triangleq \frac{d\kappa}{d\rho}$, $\kappa^\varepsilon \triangleq \kappa(\rho^\varepsilon)$, etc.
- C denotes some harmless constant which may depend on the lower bound $\underline{\rho}$ and the upper bound $\overline{\rho}$ of the initial density (see (1.13)). In some places, we shall alternately use the notation $A \lesssim B$ instead of $A \leq CB$.
- Functions of the form $\langle f \rangle_\varepsilon$ will always be viewed as the regularized functions of f , in the sense specified in Section 2.1.1 (see (2.33)).
- If X, Y are two Banach spaces, then the notation $X \hookrightarrow Y$ means that space X can be imbedded into space Y continuously while $X \hookrightarrow\hookrightarrow Y$ says that the embedding from X to Y is furthermore compact.
- We write $u_n \rightarrow u$ in some Banach space X to represent the strong convergence of the sequence $\{u_n\}_n$ to u in space X such that $\|u_n - u\|_X \rightarrow 0$, while $u_n \rightharpoonup u$ and $u_n \xrightarrow{*} u$ in X mean that $\{u_n\}_n$ converges to u in the associated weak and weak-* topology of space X respectively.
- The index p' denotes the conjugate of p such that $\frac{1}{p} + \frac{1}{p'} = 1$.
- If X is a Banach space, $T > 0$ and $p \in [1, +\infty]$, then $L_T^p(X)$ stands for the set of Lebesgue measurable functions f from $[0, T)$ to X such that $t \mapsto \|f(t)\|_X$ belongs to $L^p([0, T))$. If $T = +\infty$, then the space is merely denoted by $L^p(X)$. Finally, if I is some interval of \mathbb{R} then the notation $\mathcal{C}(I; X)$ stands for the set of continuous functions from I to X .
- $L_w^2(X)$ denotes the Lebesgue space $L^2(X)$ endowed with weak topology.
- We shall keep the same notation X to designate vector-fields with components in X .

The rest of the paper unfolds as follows. The next section is devoted to proving Theorem 1.1 and Theorem 1.2 whereas the proof of Theorem 1.4 is left in the third section.

2 Global existence of the weak solution

In this section we will prove the global-in-time existence of weak solutions, i.e. Theorem 1.1 and Theorem 1.2. The first paragraph is devoted to the case when the initial density ρ_0 satisfies $\rho_0 - 1 \in L^2(\mathbb{R}^d)$ and the second, is to the case where $\rho_0 - 1 \in H^1(\mathbb{R}^2)$ or $H^1(\mathbb{R}^3)$.

2.1 The case with the density of lower regularity

In this subsection we will prove Theorem 1.1 in two steps. The first step, i.e. Section 2.1.1, is to solve the regularized system of the Cauchy problem (1.7)-(1.13) while the second subsection, is devoted to show that the convergent limit of the obtained regular solution sequence is indeed a weak solution of this Cauchy problem.

2.1.1 Regularized system

In this subsection we will consider the regularized system of (1.7)-(1.13). More precisely, let us fix a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that

$$\text{Supp } \varphi \in B(0, 1), \quad 0 \leq \varphi \leq 1, \quad \int_{\mathbb{R}^d} \varphi = 1,$$

and consequently define a sequence of functions $\{\varphi_\varepsilon\}_\varepsilon$ such that $\varphi_\varepsilon(x) = \varepsilon^{-d} \varphi(\varepsilon^{-1}x)$, $\forall x \in \mathbb{R}^d$. Given any $f \in \mathcal{D}'(\mathbb{R}^d)$, we set the regularized functions $\{\langle f \rangle_\varepsilon\}_\varepsilon$ as

$$\langle f \rangle_\varepsilon \triangleq \varphi_\varepsilon * f. \quad (2.33)$$

Now we regularize the Cauchy problem as following²

$$\left\{ \begin{array}{ll} \partial_t \rho + \text{div}(\rho \langle u \rangle_\varepsilon) - \text{div}(\langle \kappa \rangle_\varepsilon \nabla \rho) & = 0, \\ \partial_t(\rho u) + \text{div}((\rho \langle u \rangle_\varepsilon - \langle \kappa \rangle_\varepsilon \nabla \rho) \otimes u) - \text{div}(2\mu A u) + \nabla \pi & = 0, \\ \text{div } u & = 0, \\ \rho|_{t=0} & = \langle \rho_0 \rangle_\varepsilon, \\ u|_{t=0} & = \langle u_0 \rangle_\varepsilon. \end{array} \right. \quad (2.34)$$

It is easy to see that if initial data (ρ_0, u_0) satisfies (1.13), then we have the following properties (keep in mind that $\varrho_0 = \rho_0 - 1$):

$$\langle \varrho_0 \rangle_\varepsilon, \langle u_0 \rangle_\varepsilon \in H^\infty, \quad \langle \varrho_0 \rangle_\varepsilon \rightarrow \varrho_0 \text{ in } L^2, \quad \langle u_0 \rangle_\varepsilon \rightarrow u_0 \text{ in } L^2, \quad 0 < \underline{\rho} \leq \langle \rho_0 \rangle_\varepsilon \leq \bar{\rho}, \quad \text{div } \langle u_0 \rangle_\varepsilon = 0.$$

In the following we will use fixed point method to solve System (2.34). More precisely, for any $T \in (0, \infty)$ fixed, we will show that the operator F from some known functions $(\tilde{\rho}, \tilde{u})$ to the solution (ρ, u) of the system (with $\tilde{\kappa} = \kappa(\tilde{\rho})$)

$$\left\{ \begin{array}{ll} \partial_t \rho + \text{div}(\rho \langle \tilde{u} \rangle_\varepsilon) - \text{div}(\langle \tilde{\kappa} \rangle_\varepsilon \nabla \rho) & = 0, \\ \partial_t(\rho u) + \text{div}((\rho \langle \tilde{u} \rangle_\varepsilon - \langle \tilde{\kappa} \rangle_\varepsilon \nabla \rho) \otimes u) - \text{div}(2\mu(\rho) A u) + \nabla \pi & = 0, \\ \text{div } u & = 0, \\ \rho|_{t=0} & = \langle \rho_0 \rangle_\varepsilon, \\ u|_{t=0} & = \langle u_0 \rangle_\varepsilon, \end{array} \right. \quad (2.35)$$

is compact in the Banach space

$$E_{R_0, T} = \left\{ (\rho, u) \mid (\rho - 1, u) \in C([0, T]; L^2) \cap L^2(0, T; H^1), \quad \text{div } u = 0 \text{ on } \mathbb{R}^d \times [0, T], \right. \\ \left. 0 < \underline{\rho} \leq \rho \leq \bar{\rho}, \quad \|(\rho - 1, u)\|_{L^\infty(0, T; L^2)}, \|(\nabla \rho, \nabla u)\|_{L^2(0, T; L^2)} \leq R_0 \right\}, \quad (2.36)$$

with R_0 depending only on the initial data, to be determined later. Let us first notice that although System (2.35) is nonlinear, after getting the solution ρ to the linear system (2.35)₁ – (2.35)₄, the equations (2.35)₂ – (2.35)₃ for the velocity u and the pressure π become linear immediately. Besides, thanks to the regularization, we will show that the solution (ρ, u) to System (2.35) belongs to a much more regular solution space than $E_{R_0, T}$, which provides F with compactity.

Now we state the well-posedness result for System (2.34):

Proposition 2.1. *For any positive time $T \in (0, +\infty)$, there exists a unique smooth solution $(\rho, u) \in E_{R_0, T}$ to System (2.34) such that $(\rho - 1, u) \in C([0, T]; H^\infty)$.*

²We notice here that we don't have to regularize the coefficient μ since the density ρ as a solution of the regularized equation (2.34)₁ is already smooth, whereas we remain the regularized form of the coefficient κ to keep uniform with (2.34)₁. This implies the uniform energy bound for u .

Proof. Throughout the proof, we will use frequently the notation C_ε to denote the constants which may depend on $\varepsilon, T, \underline{\rho}, \bar{\rho}$ and R_0 whereas the notation $C_\varepsilon(m)$, denotes those constants C_ε depending additionally on m .

We consider first the linear equation for ϱ :

$$\begin{cases} \partial_t \varrho + \langle \tilde{u} \rangle_\varepsilon \cdot \nabla \varrho - \operatorname{div}(\langle \tilde{\kappa} \rangle_\varepsilon \nabla \varrho) &= 0, \\ \varrho|_{t=0} &= \langle \varrho_0 \rangle_\varepsilon. \end{cases} \quad (2.37)$$

To solve it, we will use Friedrich's method. For any $n \in \mathbb{N}$, we define the space L_n^2 to be the closed set of L^2 functions with Fourier transform supported in the ball of center 0 and radius n and the associated orthogonal projector P_n is defined by $\widehat{P_n f}(\xi) = 1_{|\xi| \leq n} \widehat{f}(\xi)$, then we immediately get a unique solution $\varrho_n \in C([0, T]; L_n^2) \cap C^1((0, T); L_n^2)$ to the following system:

$$\begin{cases} \partial_t \varrho_n + P_n(\langle \tilde{u} \rangle_\varepsilon \cdot \nabla \varrho_n) - P_n \operatorname{div}(\langle \tilde{\kappa} \rangle_\varepsilon \nabla \varrho_n) &= 0, \\ \varrho_n|_{t=0} &= P_n \langle \varrho_0 \rangle_\varepsilon. \end{cases} \quad (2.38)$$

In fact, it is easy to see that the above equation is a linear ordinary differential equation on L_n^2 .

Now taking the $L^2(\mathbb{R}^d)$ inner product between (2.38) and ϱ_n and integrating by parts give the following a priori estimate (noticing that $\operatorname{div} \langle \tilde{u} \rangle_\varepsilon = 0$, $P_n \varrho_n = \varrho_n$ and $\langle P_n f, g \rangle_{L^2} = \langle f, P_n g \rangle_{L^2}$):

$$\frac{1}{2} \frac{d}{dt} \|\varrho_n\|_{L^2}^2 + \int_{\mathbb{R}^d} \langle \tilde{\kappa} \rangle_\varepsilon |\nabla \varrho_n|^2 = 0,$$

that is,

$$\|\varrho_n\|_{L_T^\infty(L^2)}^2 + C \|\nabla \varrho_n\|_{L_T^2(L^2)}^2 \leq \|\varrho_n|_{t=0}\|_{L^2}^2 \leq \|\varrho_0\|_{L^2}^2. \quad (2.39)$$

Similarly, we can multiply (2.38) by $\Delta \varrho_n, \Delta^2 \varrho_n, \dots$ and integrate on the whole space \mathbb{R}^d , to get

$$\|\varrho_n\|_{L_T^\infty(H^m)}^2 + \|\varrho_n\|_{L_T^2(H^{m+1})}^2 \leq C_\varepsilon(m), \quad \forall m \geq 0. \quad (2.40)$$

Hence, the fact that ϱ_n solves (2.38) implies that

$$\|\partial_t \varrho_n\|_{L_T^\infty(H^m)} \leq \|\langle \tilde{u} \rangle_\varepsilon \cdot \nabla \varrho_n\|_{L_T^\infty(H^m)} + \|\langle \tilde{\kappa} \rangle_\varepsilon \nabla \varrho_n\|_{L_T^\infty(H^{m+1})} \leq C_\varepsilon(m). \quad (2.41)$$

Thus by Inequality (2.40), Inequality (2.41) and Arzelà-Ascoli Theorem, there exists one unique $\varrho \in C([0, T]; H^\infty)$ such that for any fixed $m \geq 0$, one has a convergent subsequence $\{\varrho_{n(m)}\}$ with

$$\varrho_{n(m)} \rightarrow \varrho \text{ in } L_T^\infty(H_{\text{loc}}^m). \quad (2.42)$$

Now we rewrite (2.38) as

$$\partial_t \varrho_n + \langle \tilde{u} \rangle_\varepsilon \cdot \nabla \varrho_n - \operatorname{div}(\langle \tilde{\kappa} \rangle_\varepsilon \nabla \varrho_n) = (\operatorname{Id} - P_n) \operatorname{div}(\langle \tilde{u} \rangle_\varepsilon \varrho_n - \langle \tilde{\kappa} \rangle_\varepsilon \nabla \varrho_n). \quad (2.43)$$

Since $\forall s \in \mathbb{R}$, we have

$$\|(\operatorname{Id} - P_n)f\|_{H^s} \leq \frac{1}{n} \|f\|_{H^{s+1}}, \quad (2.44)$$

thus let $n(m)$ go to ∞ , then the above control and the convergence result (2.42) implies that the limit $\varrho \in C([0, T]; H^\infty)$ really solves (2.37) and satisfies Estimates (2.39), (2.40) and (2.41). Moreover, by maximum principle, we have

$$0 < \underline{\rho} \leq \varrho + 1 \leq \bar{\rho}. \quad (2.45)$$

Now we move to solve the following system in $C([0, T]; H^\infty)$ with $\varrho \in C([0, T]; H^\infty)$ satisfying (2.37) given by the Step 1 (which amounts to solving (2.35)₂ – (2.35)₃ – (2.35)₅):

$$\begin{cases} \rho \partial_t u + (\rho \langle \tilde{u} \rangle_\varepsilon - \langle \tilde{\kappa} \rangle_\varepsilon \nabla \rho) \cdot \nabla u - \operatorname{div}(2\mu A u) + \nabla \pi &= 0, \\ \operatorname{div} u &= 0, \\ u|_{t=0} &= \langle u_0 \rangle_\varepsilon. \end{cases} \quad (2.46)$$

We will proceed exactly as above. Firstly, we look for $u_n \in C([0, T]; L_n^2) \cap C^1((0, T); L_n^2)$ satisfying

$$\begin{cases} \partial_t u_n + P_n(Lu_n) &= 0, \\ u_n|_{t=0} &= P_n \langle u_0 \rangle_\varepsilon, \end{cases} \quad (2.47)$$

where the linear operator L is defined by³

$$Lu_n = \left(\langle \tilde{u} \rangle_\varepsilon - \langle \tilde{\kappa} \rangle_\varepsilon \rho^{-1} \nabla \rho \right) \cdot \nabla u_n + \rho^{-1} \nabla \mu \cdot Du_n - \rho^{-1} \operatorname{div}(\mu \nabla u_n) + \rho^{-1} \nabla \pi_n, \quad (2.48)$$

with $\nabla \pi_n$ satisfying

$$\operatorname{div}(\rho^{-1} \nabla \pi_n) = -\operatorname{div} \left((\langle \tilde{u} \rangle_\varepsilon - \rho^{-1} \langle \tilde{\kappa} \rangle_\varepsilon \nabla \rho) \cdot \nabla u_n + 2\mu \nabla \rho^{-1} \cdot Au_n \right). \quad (2.49)$$

We point out here that the following equality holds true:

$$\operatorname{div}(\rho^{-1} \operatorname{div}(2\mu Au)) \equiv -\operatorname{div}(2\mu \nabla \rho^{-1} \cdot Au). \quad (2.50)$$

It is easy to see that the unique map from u_n to $\nabla \pi_n$ defined by the equation (2.49) is continuous such that

$$\|\nabla \pi_n\|_{L^2} \leq C_\varepsilon \|\nabla u_n\|_{L^2} \leq C_\varepsilon(n) \|u_n\|_{L_n^2}. \quad (2.51)$$

Therefore the linear map $u_n \mapsto P_n(Lu_n)$ is continuous on L_n^2 which, ensures one unique solution $u_n \in C([0, T]; L_n^2) \cap C^1((0, T); L_n^2)$ of System (2.47).

Now we are at the point to get the uniform estimates for u_n to show the convergence. Taking the L^2 inner product between (2.47) and u_n implies that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |u_n|^2 + \int_{\mathbb{R}^d} \mu \rho^{-1} |\nabla u_n|^2 \leq C_\varepsilon \|\nabla u_n\|_{L^2} \|u_n\|_{L^2}.$$

Thus by Hölder's inequality the following uniform estimate for u_n holds:

$$\|u_n\|_{L_T^\infty(L^2)} + \|\nabla u_n\|_{L_T^2(L^2)} \leq C_\varepsilon \|u_0\|_{L^2}. \quad (2.52)$$

Since by induction we have from (2.49) that

$$\|\nabla \pi_n\|_{H^m} \leq C_\varepsilon(m) \|\nabla u_n\|_{H^m},$$

we can multiply (2.47) by $\Delta^m u_n$, with $m \geq 0$ to derive

$$\|u_n\|_{L_T^\infty(H^m)} + \|\partial_t u_n\|_{L_T^\infty(H^m)} + \|\nabla \pi_n\|_{L_T^\infty(H^m)} \leq C_\varepsilon(m), \quad \forall m \geq 0. \quad (2.53)$$

Hence there exist $u \in C([0, T]; H^\infty)$ and $\nabla \pi \in L^\infty((0, T); H^\infty)$ which is given by (2.49) with u_n replaced by u , such that $u(0) = \langle u_0 \rangle_\varepsilon$ and for any fixed $m \geq 0$, there exists a subsequence $u_{n(m)}$, $\nabla \pi_{n(m)}$ verifying

$$u_{n(m)} \rightarrow u \text{ in } L_T^\infty(H_{\text{loc}}^m), \quad \nabla \pi_{n(m)} \rightharpoonup \nabla \pi \text{ in } L_T^\infty(H_{\text{loc}}^m).$$

Moreover, (2.44) entails

$$\partial_t u + Lu = 0. \quad (2.54)$$

Applying the divergence operator div to it yields⁴

$$\partial_t(\operatorname{div} u) - \operatorname{div}(\mu \rho^{-1} \nabla \operatorname{div} u) = 0. \quad (2.55)$$

³We multiply (2.46)₁ by ρ^{-1} and rewrite the quantity $-\rho^{-1} \operatorname{div}(2\mu Au)$ into a summation of one 2-order term and one 1-order term by use of $\operatorname{div} u = 0$.

⁴By Definition (2.48) of the operator L , Equation (2.49) of π and Equality (2.50), we have $\operatorname{div} Lu = \operatorname{div}(\rho^{-1} \nabla \mu \cdot Du - \rho^{-1} \operatorname{div}(\mu \nabla u) + \rho^{-1} \operatorname{div}(2\mu Au)) = \operatorname{div}(\rho^{-1} \nabla \mu \cdot Du - \rho^{-1} \operatorname{div}(\mu Du)) = -\operatorname{div}(\mu \rho^{-1} \nabla \operatorname{div} u)$.

Thus the parabolic equation (2.55) ensures that $\operatorname{div} u = 0$. Therefore u truly solves (2.46). Furthermore, we take the inner product between (2.46) and u , issuing the Energy Equality (1.12) by use of Equation (2.37) for ρ , which together with the identity $\|Au\|_{L^2} = \|\nabla u\|_{L^2}$ (by $\operatorname{div} u = 0$) entails

$$\|u\|_{L^\infty([0,t];L^2)} + \|\nabla u\|_{L^2([0,t];L^2)} \leq C(\underline{\rho}, \bar{\rho}) \|u_0\|_{L^2}, \quad \forall t > 0. \quad (2.56)$$

At last, noticing (2.39) and (2.56), we just have to choose R_0 depending on $\|\varrho_0\|_{L^2}, \|u_0\|_{L^2}, \underline{\rho}, \bar{\rho}$ such that the operator $F : (\tilde{\rho}, \tilde{u}) \mapsto (\rho, u)$ maps from $E_{R_0,T}$ to $E_{R_0,T}$. Furthermore, the boundedness (2.40), (2.41) and (2.53) ensures that

$$F : E_{R_0,T} \mapsto E_{R_0,T} \cap \left\{ (\rho, u) \mid \|(\varrho, \partial_t \rho, u, \partial_t u)\|_{L_T^\infty(H^m)} \leq C_\varepsilon(m), \forall m \geq 0 \right\},$$

which implies that F is compact in $C([0,T]; H_{\text{loc}}^m)$ for all $m \geq 0$. Hence there exists one unique fixed point $(\varrho, u) \in C([0,T]; H^\infty)$ of the operator F in $E_{R_0,T}$ which is also the solution to System (2.34). This ends the proof of Proposition 2.1. \square

Remark 2.1. Since in Banach space $E_{R_0,T}$, the bound R_0 is independent of the time T , Proposition 2.1 actually permits that for any ε , there exists a unique globally-in-time existing smooth solution $(\varrho^\varepsilon, u^\varepsilon) \in C([0, +\infty); H^\infty)$ to System (2.34) such that $\forall \varepsilon > 0$,

$$\rho^\varepsilon \in [\underline{\rho}, \bar{\rho}], \quad \|\varrho^\varepsilon\|_{L^\infty(L^2)} + \|\nabla \rho^\varepsilon\|_{L^2(L^2)} \leq C \|\varrho_0\|_{L^2(\mathbb{R}^d)}, \quad \|u^\varepsilon\|_{L^\infty(L^2)} + \|\nabla u^\varepsilon\|_{L^2(L^2)} \leq C \|u_0\|_{L^2(\mathbb{R}^d)}, \quad (2.57)$$

with C being a constant depending only on $\underline{\rho}, \bar{\rho}$.

2.1.2 The convergence to a weak solution

We now are at the point to show that the solution sequence given by Section 2.1.1 converges to a weak solution to Cauchy problem (1.7)-(1.13). The smoothing effect on both variables is useful to use the compactness methods and the strategy is hence quite standard. So let's just sketch the proof.

By Remark 2.1 at the end of Section 2.1.1, we may assume that there exist subsequences $\{\rho^{\varepsilon_n}\}_n$ and $\{u^{\varepsilon_n}\}_n$ of the solution sequences $\{\rho^\varepsilon\}_\varepsilon$ and $\{u^\varepsilon\}_\varepsilon$ respectively such that

$$\varrho^{\varepsilon_n} \xrightarrow{*} \varrho \text{ in } L^\infty(L^2 \cap L^\infty), \quad u^{\varepsilon_n} \xrightarrow{*} u \text{ in } L^\infty(L^2), \quad \nabla \varrho^{\varepsilon_n} \rightharpoonup \nabla \varrho, \quad \nabla u^{\varepsilon_n} \rightharpoonup \nabla u \text{ in } L^2(L^2),$$

with the limit (ρ, u) verifying Estimate (2.57) too.

Thanks to the uniform bound on $\nabla \rho$ and by use of the regularized system (2.34)₁ – (2.34)₄, we can easily use a diagonal process to show that ρ solves Equation (1.7)₁ in distribution sense. For example, there exists a subsequence, still denoted by κ^{ε_n} , such that $\langle \kappa^{\varepsilon_n} \rangle_{\varepsilon_n} - \kappa \rightarrow 0, a.e.$ Thus $\langle \kappa^{\varepsilon_n} \rangle_{\varepsilon_n} \nabla \rho^{\varepsilon_n} \rightharpoonup \kappa \nabla \rho$ in $L^2(L^2(\mathbb{R}^d))$.

The above bound (2.57) furthermore ensures that Equation (1.7)₁ holds in $L_{\text{loc}}^2(H^{-1})$. Thus, we can test it by the solution $\varrho \in L^2(H^1)$ itself such that Energy Equality (1.11) hold for ϱ^ε and ϱ both (notice that $\operatorname{div} u = 0$ in $L_w^2([0, \infty) \times \mathbb{R}^d)$), i.e. for all $t \in [0, \infty)$,

$$\frac{1}{2} \|\varrho^\varepsilon(t)\|_{L^2}^2 + \|\langle \kappa^\varepsilon \rangle_\varepsilon^{\frac{1}{2}} \nabla \varrho^\varepsilon\|_{L_t^2(L^2)}^2 = \frac{1}{2} \|\langle \varrho_0 \rangle_\varepsilon\|_{L^2}^2, \quad \frac{1}{2} \|\varrho(t)\|_{L^2}^2 + \|\kappa^{\frac{1}{2}} \nabla \varrho\|_{L_t^2(L^2)}^2 = \frac{1}{2} \|\varrho_0\|_{L^2}^2. \quad (2.58)$$

Now we consider the quantity

$$\frac{1}{2} \|\varrho^{\varepsilon_n}(t) - \varrho(t)\|_{L^2(\mathbb{R}^d)}^2 + \|\langle \kappa^{\varepsilon_n} \rangle_{\varepsilon_n}^{\frac{1}{2}} \nabla \varrho^{\varepsilon_n} - \kappa^{\frac{1}{2}} \nabla \varrho\|_{L_t^2(L^2)}^2,$$

which by Energy Equality (2.58), is equal to

$$\frac{1}{2} \|\langle \varrho_0 \rangle_{\varepsilon_n}\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2} \|\varrho_0\|_{L^2(\mathbb{R}^d)}^2 - \left\langle \varrho^{\varepsilon_n}(t), \varrho(t) \right\rangle_{L^2(\mathbb{R}^d)} - 2 \left\langle \langle \kappa^{\varepsilon_n} \rangle_{\varepsilon_n}^{\frac{1}{2}} \nabla \varrho^{\varepsilon_n}, \kappa^{\frac{1}{2}} \nabla \varrho \right\rangle_{L_t^2(L^2)}.$$

Since we have also $\langle \kappa^{\varepsilon_n} \rangle_{\varepsilon_n}^{\frac{1}{2}} \nabla \varrho^{\varepsilon_n} \rightharpoonup \kappa^{\frac{1}{2}} \nabla \varrho$ in $L^2(L^2)$, the above quantity converges to

$$\|\varrho_0\|_{L^2}^2 - \|\varrho(t)\|_{L^2}^2 - 2\|\kappa^{\frac{1}{2}} \nabla \varrho\|_{L_t^2(L^2)}^2 = 0.$$

This implies that

$$\varrho^{\varepsilon_n} \rightarrow \varrho \text{ in } L^\infty(L^2) \quad \text{and} \quad \langle \kappa^{\varepsilon_n} \rangle_{\varepsilon_n}^{\frac{1}{2}} \nabla \varrho^{\varepsilon_n} \rightarrow \kappa^{\frac{1}{2}} \nabla \varrho \text{ in } L^2(L^2).$$

Thus, by the boundedness of $\|\varrho^{\varepsilon_n}\|_{L^\infty([0,\infty)\times\mathbb{R}^d)}$, we have $\varrho^{\varepsilon_n} \rightarrow \varrho$ in $L^\infty(L^p)$, $\forall p \in [2, \infty)$. Therefore, $\varrho|_{t=0} = \varrho_0$ in L^p for all $p \in [2, \infty)$.

The following statement concerning the velocity u follows exactly Proof of Theorem 2.4 in the P.-L. Lions's book [36]. Let us recall it briefly for the reader's convenience. Let us first observe that the Sobolev embedding ensures that $\{u^\varepsilon\}_\varepsilon$ is bounded in $L_T^\infty(L^2) \cap L_T^2(L^{\frac{2d}{d-2}})$ (or $L_T^2(L^p)$ with $p \in [2, \infty)$ if $d = 2$) for any positive finite time T and hence

$$\begin{aligned} \{\langle u^\varepsilon \rangle_\varepsilon \otimes u^\varepsilon\} &\text{ is bounded in } L_T^\infty(L^1) \cap L_T^2(L^{\frac{d}{d-1}}) \text{ (or } L_T^2(L^p) \text{ with } p \in [1, 2) \text{ if } d = 2), \\ \text{and } \{\nabla \rho^\varepsilon \otimes u^\varepsilon\} &\text{ is bounded in } L_T^2(L^1) \cap L_T^1(L^{\frac{d}{d-1}}) \text{ (or } L_T^1(L^p) \text{ with } p \in [1, 2) \text{ if } d = 2). \end{aligned}$$

Therefore, in view of the equation (2.34)₂ for u^ε , there exist constants $p \in (2, \infty)$, $m > 1^5$ and M depending on T, R_0 such that for all divergence-free function $\phi \in L_T^p(H^m)$ we have

$$\left| \left\langle \partial_t(\rho^\varepsilon u^\varepsilon), \phi \right\rangle_{\mathcal{D}', \mathcal{D}} \right| = \left| \left\langle -(\rho^\varepsilon \langle u^\varepsilon \rangle_\varepsilon - \langle \kappa^\varepsilon \rangle_\varepsilon \nabla \rho^\varepsilon) \otimes u^\varepsilon + 2\mu^\varepsilon A u^\varepsilon, \nabla \phi \right\rangle_{\mathcal{D}', \mathcal{D}} \right| \leq M \|\phi\|_{L_T^p(H^m)}. \quad (2.59)$$

Let us notice that the Leray projector $\mathcal{P} := \text{Id} + \nabla(-\Delta)^{-1} \text{div}$ is bounded on each Sobolev space H^s . Hence from (2.59), we actually have $\partial_t(\mathcal{P}(\rho^\varepsilon u^\varepsilon))$ is bounded in $L_T^{p'}(H^{-m})$. Since $\rho^{\varepsilon_n} u^{\varepsilon_n}$ converges weakly to ρu , the boundedness of $\{\rho^\varepsilon u^\varepsilon\}$ in $L_T^\infty(L^2)$ implies the existence of one convergent subsequence (still denoted by $\rho^{\varepsilon_n} u^{\varepsilon_n}$) such that $\mathcal{P}(\rho^{\varepsilon_n} u^{\varepsilon_n}) \rightarrow \mathcal{P}(\rho u)$ in $C([0, T]; L_w^2)$. Hence, we have for any $t > 0$,

$$\int_0^t \int_{\mathbb{R}^d} \rho^{\varepsilon_n} |u^{\varepsilon_n}|^2 = \int_0^t \langle \mathcal{P}(\rho^{\varepsilon_n} u^{\varepsilon_n}), u^{\varepsilon_n} \rangle \rightarrow \int_0^t \langle \mathcal{P}(\rho u), u \rangle = \int_0^t \int_{\mathbb{R}^d} \rho |u|^2.$$

Thus $\rho^{\varepsilon_n} u^{\varepsilon_n} \rightarrow \rho u$ and $u^{\varepsilon_n} \rightarrow u$ in $L_{\text{loc}}^2(L^2)$. Moreover, we apply Theorem 2.2 in [36] to get $u \in C([0, \infty); L_w^2)$. It is easy to find that

$$\rho^{\varepsilon_n} \langle u^{\varepsilon_n} \rangle_{\varepsilon_n} \otimes u^{\varepsilon_n} \rightarrow \rho u \otimes u \text{ in } L_T^2(L^1), \quad \mu^{\varepsilon_n} A u^{\varepsilon_n} \rightharpoonup \mu A u \text{ in } L_T^2(L^2),$$

and $\langle \kappa^{\varepsilon_n} \rangle_{\varepsilon_n} \nabla \rho^{\varepsilon_n} \otimes u^{\varepsilon_n} \rightarrow \kappa \nabla \rho \otimes u$ in $L_T^1(L^1)$. Thus observing Equation (2.34)₂, there exists some distribution of gradient form $\nabla \pi$ such that Equation (1.7)₂ holds for the above limit u at least in distribution sense and hence (1.15) holds. According to Theorem 2.2 in [36], the conclusion $u \in C(L_w^2)$ results from the initial assumption $\text{div } u_0 = 0$.

This completes the proof of Theorem 1.1.

2.2 The case with the density of higher regularity

In this section we will tackle the case with smoother density. It is easy to see that proving Theorem 1.2 equals to proving the following:

Let $d = 2, 3$. For any initial data (ρ_0, u_0) such that

$$0 < \underline{\rho} \leq \rho_0 \leq \overline{\rho}, \quad \rho_0 - 1 \in H^1(\mathbb{R}^d), \quad u_0 \in L^2(\mathbb{R}^d), \quad \text{div } u_0 = 0, \quad (2.60)$$

⁵We notice that $H^{m_1} \hookrightarrow W^{m_2, q}$ if $m_1 - \frac{d}{2} \geq m_2 - \frac{d}{q}$, $q \geq 2$.

System (1.7) has a weak solution (ρ, u) satisfying (1.22) and (1.23).

Thus in the first paragraph of this section, by establishing a new a priori estimate in smoother functional space in dimension 2, we deduce that if (2.60) holds for the initial data, then so do the weak solutions (ρ, u) got in the last section. However in dimension 3, we will reprove the existence of weak solutions by regularizing the system in two levels which, ensures us to get the uniform estimate (1.19) when the transport velocity is still regularized. This will be done in the second paragraph.

2.2.1 2D case

We will establish two lemmas (Lemma 2.1 and Lemma 2.2), in order to show that the global weak solutions $(\varrho, u, \nabla\pi) = (\rho - 1, u, \nabla\pi)$ given by Theorem 1.1 but with smoother initial density $\varrho_0 \in H^1(\mathbb{R}^2)$, are bounded only by initial data in the Banach space $X_2(T) \times X_1(T) \times X_{-1}(T)$ for all $T \in [0, \infty]$ in dimension 2 with

$$\begin{aligned} X_2(T) &\triangleq \{\varrho \in L_T^\infty(H^1(\mathbb{R}^2)) \mid \nabla\varrho \in L_T^2(H^1(\mathbb{R}^2)), \quad 0 < \underline{\varrho} \leq \varrho(t) + 1 \leq \overline{\varrho}, \forall t \in [0, T]\}, \\ X_1(T) &\triangleq \{u \in L_T^\infty(L^2(\mathbb{R}^2)) \mid \nabla u \in L_T^2(L^2(\mathbb{R}^2))\}, \\ X_{-1}(T) &\triangleq (X_1(T))' : \text{ the dual of the Banach space } X_1(T). \end{aligned}$$

However, in order to prove uniqueness and stability, it is not enough to just consider the solutions in $X_2(T) \times X_1(T) \times X_{-1}(T)$, since $H^1(\mathbb{R}^2)$ can not be embedded into $L^\infty(\mathbb{R}^2)$, which is needed in estimating the nonlinear terms. Therefore we will consider the initial data in the critical Besov spaces, in order to get a unique global strong solution. This will be done in next section.

Throughout this section, we will use frequently (explicitly or implicitly) the following Gagliardo-Nirenberg inequality in dimension 2:

$$\|f\|_{L^4(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}^{1/2} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{1/2}. \quad (2.61)$$

We notice that by (2.61), if $\varrho \in X_2(T)$, then the mapping $h \mapsto f(\varrho)h$ is an isomorphism on $X_1(T)$ (and hence $X_{-1}(T)$) for any diffeomorphism f from $[\underline{\varrho}, \overline{\varrho}]$ to \mathbb{R} . In fact, we have

$$\|f(\varrho)h\|_{X_1(T)} \leq C(f, \|\varrho\|_{L_T^\infty(L^\infty)}) (\|h\|_{X_1(T)} + \|\nabla\varrho\|_{L_T^4(L^4)} \|h\|_{L_T^4(L^4)}) \leq C(f, \|\varrho\|_{X_2(T)}) \|h\|_{X_1(T)}.$$

We now prove first that $\nabla\pi \in X_{-1}(T)$ if $\varrho \in X_2(T), u \in X_1(T)$. Indeed, we just have to show that the convergent regular sequence $\nabla\pi^{\varepsilon_n}$ which are solutions of the following equation (see Equation (2.49))

$$\operatorname{div}(\rho^{-1}\nabla\pi) = -\operatorname{div}((\langle u \rangle_\varepsilon - \rho^{-1}\langle \kappa \rangle_\varepsilon \nabla\varrho) \cdot \nabla u + 2\mu\nabla\rho^{-1} \cdot Au), \quad (2.62)$$

are uniformly bounded in $X_{-1}(T)$. Hence we introduce the following lemma:

Lemma 2.1. *Assume that the smooth triplet $(\varrho, u, \nabla\pi)$ satisfies Equation (2.62), then there exists one constant C_1 depending only on $\|\varrho\|_{X_2(T)}, \|u\|_{X_1(T)}$ such that*

$$\|\nabla\pi\|_{X_{-1}(T)} \leq C_1. \quad (2.63)$$

Proof. The proof is very similar to the proof of Lemma 2.1 in [36]. Given any $h \in X_1(T)$, we first claim that we have the decomposition $h = h_1 + h_2$ such that

$$\mathcal{R} \wedge (\rho h_1) = 0, \quad \operatorname{div} h_2 = 0, \quad \|h_i\|_{X_1(T)} \leq C(\|\varrho\|_{X_2(T)}) \|h\|_{X_1(T)}, \quad i = 1, 2, \quad (2.64)$$

where $\mathcal{R}_i = (-\Delta)^{-1/2} \frac{\partial}{\partial x_i}$, $i = 1, 2$ denotes the usual Riesz transform and $f \wedge g := f_1 g_2 - f_2 g_1$.

In fact, we decompose the function $\rho h \in X_1(T)$ such that

$$\rho h = \tilde{h}_1 + \tilde{h}_2 = -\mathcal{R}(\mathcal{R} \cdot (\rho h)) + (\operatorname{Id} + \mathcal{R}(\mathcal{R} \cdot))(\rho h), \quad \mathcal{R} \wedge \tilde{h}_1 = 0, \quad \mathcal{R} \cdot \tilde{h}_2 = 0,$$

and

$$\|\tilde{h}_i\|_{X_1(T)} \leq C\|\rho h\|_{X_1(T)} \leq C(\|\varrho\|_{X_2(T)})\|h\|_{X_1(T)}, \quad i = 1, 2.$$

To prove (2.64) then amounts to searching for a unique function $\mathcal{R}v \in X_1(T)$ such that

$$h_1 = \rho^{-1}\tilde{h}_1 - \rho^{-1}\mathcal{R}v, \quad h_2 = \rho^{-1}\tilde{h}_2 + \rho^{-1}\mathcal{R}v,$$

with

$$\operatorname{div}(\rho^{-1}\nabla V + \rho^{-1}\tilde{h}_2) = 0, \quad V = (-\Delta)^{-1/2}v. \quad (2.65)$$

According to Section 3 in [19], Equation (2.65) admits one unique solution $\nabla V = \mathcal{R}v$ such that

$$\|\mathcal{R}v\|_{L_T^\infty(L^2)} \leq C\|\tilde{h}_2\|_{L_T^\infty(L^2)} \leq C\|\rho h\|_{L_T^\infty(L^2)} \leq C(\|\varrho\|_{X_2(T)})\|h\|_{L_T^\infty(L^2)}.$$

Now we take the derivative ∇ to (2.65) to arrive at

$$\operatorname{div}(\rho^{-1}\nabla^2 V + \nabla V \otimes \nabla \rho^{-1} + (\nabla(\rho^{-1}\tilde{h}_2))^T) = 0,$$

which similarly gives

$$\|\nabla \mathcal{R}v\|_{L_T^2(L^2)} \leq C(\|\varrho\|_{X_2(T)})\left(\|\nabla V\|_{L_T^4(L^4)}\|\nabla \rho^{-1}\|_{L_T^4(L^4)} + \|\rho^{-1}\tilde{h}_2\|_{X_1(T)}\right).$$

Thus by (2.61) and Young's inequality (2.64) follows.

By the decomposition (2.64), we have for any $h \in X_1(T)$,

$$\langle \nabla \pi, h \rangle_{X_{-1}(T), X_1(T)} = \langle \nabla \pi, h_1 \rangle = \langle \rho^{-1}\nabla \pi, \rho h_1 \rangle.$$

Therefore Equation (2.62) for $\nabla \pi$ yields

$$\begin{aligned} |\langle \nabla \pi, h \rangle| &= \left| \left\langle (\langle u \rangle_\varepsilon - \rho^{-1}\langle \kappa \rangle_\varepsilon \nabla \rho) \cdot \nabla u + 2\mu \nabla \rho^{-1} \cdot \nabla u, \rho h_1 \right\rangle \right| \\ &\leq C(\|u\|_{L_T^4(L^4)} + \|\nabla \rho\|_{L_T^4(L^4)})\|\nabla u\|_{L_T^2(L^2)}\|\rho h_1\|_{X_1(T)} \\ &\leq C(\|\varrho\|_{X_2(T)}, \|u\|_{X_1(T)})\|h_1\|_{X_1(T)}, \end{aligned}$$

which gives the lemma by (2.64). \square

Remark 2.2. We point out here that, $L_T^2(H^{-1}(\mathbb{R}^2)) \subset X_{-1}(T)$ by definition and $\Delta u \in L_T^2(H^{-1}(\mathbb{R}^2))$ since $\nabla u \in L_T^2(L^2(\mathbb{R}^2))$. But since the “low” regularity $\varrho \in X_2(T)$ can not permit that $h \mapsto \rho h$ is an isomorphism on $L_T^2(H^1(\mathbb{R}^2))$ due to a lack of control on $\|\nabla \rho \otimes h\|_{L_T^2(L^2)}$, it is not clear that $\|\nabla \pi\|_{L_T^2(H^{-1}(\mathbb{R}^2))}$ is bounded in Lemma 2.1. Even if $\rho \equiv 1$, that is in the classical incompressible Navier-Stokes equation case, it is well-known that the pressure term $\nabla \pi$ can be bounded in $L_T^2(\mathcal{M}(\mathbb{R}^2))$ with $\mathcal{M}(\mathbb{R}^2)$ the measure space, denoting the dual space of $C_0(\mathbb{R}^2)$.

It is easy to check that the following lemma concerning the unknowns $\rho, u, \nabla \pi$ immediately yields Theorem 1.2 when $d = 2$, which is omitted here.

Lemma 2.2. Let $d = 2$. For any initial data (ρ_0, u_0) satisfying (2.60), the global weak solution $(\varrho, u, \nabla \pi)$ given by Theorem 1.1 satisfies

$$\|\varrho\|_{X_2(T)} + \|u\|_{X_1(T)} + \|\nabla \pi\|_{X_{-1}(T)} + \|\partial_t \rho\|_{L_T^2(L^2)} + \|\partial_t u\|_{X_{-1}(T)} \leq C_2, \quad (2.66)$$

with C_2 depending only on $\underline{\rho}, \bar{\rho}, \|\varrho_0\|_{H^1(\mathbb{R}^2)}, \|u_0\|_{L^2(\mathbb{R}^2)}$. Moreover, Energy Equality (1.12) holds and $\varrho \in C([0, \infty); H^1(\mathbb{R}^2))$, $u \in C([0, \infty); L^2(\mathbb{R}^2))$.

Proof. By Theorem 1.1, Equation (1.7)₁ and Lemma 2.1, to prove (2.66) it rests to prove $\varrho \in X_2(T)$ and $\partial_t u \in X_{-1}(T)$, which can be assumed to be right a priori. In fact, since if $u, \nabla \rho \in X_1(T)$, then $\rho u, \kappa \nabla \rho \in X_1(T)$ by (2.61) and hence

$$\|\partial_t \rho\|_{L_T^2(L^2(\mathbb{R}^2))} = \|-\operatorname{div}(\rho u - \kappa \nabla \rho)\|_{L_T^2(L^2(\mathbb{R}^2))} \leq C(\|\varrho\|_{X_2(T)}, \|u\|_{X_1(T)}). \quad (2.67)$$

Let us deal with ϱ first. Set K to be an antiderivative of κ such that $K(1) = 0$. Since $\nabla K = \kappa \nabla \rho \in X_1(T)$, then $K \in L_T^\infty(H^1 \cap L^\infty)$ and $\nabla K \in L_T^2(H^1)$. Multiplying (1.7)₁ by $\kappa = \kappa(\rho)$ yields

$$\partial_t K + u \cdot \nabla K - \kappa \Delta K = 0 \text{ in } L_T^2(L^2). \quad (2.68)$$

Taking the $L^2(\mathbb{R}^2)$ inner product with ΔK issues

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla K|^2 - \int_{\mathbb{R}^2} u \cdot \nabla K \Delta K + \int_{\mathbb{R}^2} \kappa |\Delta K|^2 = 0. \quad (2.69)$$

For any $\varepsilon > 0$, we have from Inequality (2.61) and Young's Inequality

$$\left| \int_{\mathbb{R}^2} u \cdot \nabla K \Delta K \right| \leq \|u\|_{L^4} \|\nabla K\|_{L^4} \|\Delta K\|_{L^2} \leq \varepsilon \|\Delta K\|_{L^2}^2 + C_\varepsilon \|u\|_{L^4}^4 \|\nabla K\|_{L^2}^2,$$

for some constant C_ε . Let us choose sufficiently small ε , then we have shown (by (1.16))

$$\sup_{0 \leq t \leq T} \|\nabla K\|_{L^2}^2 + \int_0^T \|\Delta K\|_{L^2}^2 \leq e^{C \int_0^T \|u\|_{L^4}^4} C \|\nabla \varrho_0\|_{L^2}^2 \leq e^{C \|u_0\|_{L^2}^4} C \|\nabla \varrho_0\|_{L^2}^2, \quad \forall T \geq 0. \quad (2.70)$$

It is also easy to see from (2.69) that $\nabla K \in C([0, \infty); L^2)$. Thus $\varrho \in C([0, \infty); H^1)$.

Now since $\Delta K = \kappa \Delta \varrho + \nabla \kappa \cdot \nabla \varrho$, we have

$$\|\Delta \varrho\|_{L^2} \leq C(\|\Delta K\|_{L^2} + \|\nabla K\|_{L^4}^2) \leq C\|\Delta K\|_{L^2}(1 + \|\nabla K\|_{L^2}^2),$$

which already gives the estimate for ϱ :

$$\sup_{0 \leq t \leq T} \|\nabla \varrho(t)\|_{L^2}^2 + \int_0^T \|\nabla^2 \varrho\|_{L^2}^2 \leq e^{C \|u_0\|_{L^2}^4} C \|\nabla \varrho_0\|_{L^2}^2 (1 + \|\nabla \varrho_0\|_{L^2}^2), \quad \forall T \geq 0. \quad (2.71)$$

Now we turn to the equation for u . It is easy to find that

$$\partial_t u = -\rho^{-1} \left(\partial_t \rho u + \operatorname{div}((\rho u - \kappa \nabla \rho) \otimes u) - \operatorname{div}(2\mu A u) + \nabla \pi \right).$$

Thus for any $h \in X_1(T)$, we have by (2.67)

$$\begin{aligned} |\langle \partial_t u, h \rangle_{X_{-1}(T), X_1(T)}| &\leq \|\partial_t \rho\|_{L_T^2(L^2)} \|u\|_{L_T^4(L^4)} \|\rho^{-1} h\|_{L_T^4(L^4)} + |\langle \nabla \pi, \rho^{-1} h \rangle| \\ &\quad + \|(\rho u - \kappa \nabla \rho) \otimes u - 2\mu A u\|_{L_T^2(L^2)} \|\nabla(\rho^{-1} h)\|_{L_T^2(L^2)} \\ &\leq C(\|\varrho\|_{X_2(T)}, \|u\|_{X_1(T)}, \|\nabla \pi\|_{X_{-1}(T)}) \|h\|_{X_1(T)}. \end{aligned}$$

Hence (2.66) follows and (1.7)₂ holds in $X_{-1}(T)$.

In order to show the Energy Equality (1.12), we take the $\langle \cdot, \cdot \rangle_{X_{-1}(T), X_1(T)}$ inner product between Equation (1.7)₂ and u to arrive at (notice (1.7)₁ holding in $L_T^2(L^2)$)

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \rho |u|^2 + 2 \int_{\mathbb{R}^2} \mu |A u|^2 = 0,$$

which gives (1.12) immediately. This implies $u \in C([0, \infty); L^2)$. \square

Remark 2.3. Here we have to consider the function K instead of directly the density ρ , in order to get estimates on $\|\nabla \rho\|_{L_T^\infty(L^2) \cap L_T^2(H^1)}$. In fact, if we directly take the derivative on (1.7)₁, then whether the quantity $\nabla \rho \Delta \rho$ issuing from the divergence term $\operatorname{div}(\kappa \nabla \rho)$ can be killed by the “good” term $\Delta \nabla \rho$ is not clear.

2.2.2 3D case

Unlike the last paragraph, we do not have Inequality (2.61) in dimension 3. Thus $u \notin L^4(L^4(\mathbb{R}^3))$ and hence the quantity $\int_{\mathbb{R}^3} u \cdot \nabla K \Delta K$ in (2.69) doesn't make sense. However, inspired by the computations before Theorem 1.2, we can show first the uniform bounds as (1.19) for the regular approximated solutions ρ^ε and then, by lower semi-continuity of the L^2 -norm, it holds for the weak solution ρ .

More precisely, for any $\varepsilon > 0$ and any $\delta > 0$, we will consider the following regularized system of Cauchy problem (1.7)-(2.60), instead of System (2.34)⁶:

$$\left\{ \begin{array}{ll} \partial_t \rho + \operatorname{div}(\rho \langle u \rangle_\varepsilon) - \operatorname{div}(\langle \kappa \rangle_\delta \nabla \rho) & = 0, \\ \partial_t(\rho u) + \operatorname{div}\left((\rho \langle u \rangle_\varepsilon - \langle \kappa \rangle_\delta \nabla \rho) \otimes u\right) - \operatorname{div}(2\mu A u) + \nabla \pi & = 0, \\ \operatorname{div} u & = 0, \\ (\rho, u)|_{t=0} & = (\langle \rho_0 \rangle_\varepsilon, \langle u_0 \rangle_\varepsilon). \end{array} \right. \quad (2.72)$$

By view of Subsection 2.1.1, the above system has a unique solution $(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta})$ with $\rho^{\varepsilon, \delta} - 1, u^{\varepsilon, \delta} \in C([0, \infty); H^\infty)$, such that Estimate (2.57) holds uniformly in ε and δ .

Following Subsection 2.1.2, it is easy to find that there exists a global-in-time weak solution $(\rho^\varepsilon, u^\varepsilon)$ to the following system

$$\left\{ \begin{array}{ll} \partial_t \rho + \operatorname{div}(\rho \langle u \rangle_\varepsilon) - \operatorname{div}(\kappa \nabla \rho) & = 0, \\ \partial_t(\rho u) + \operatorname{div}\left((\rho \langle u \rangle_\varepsilon - \kappa \nabla \rho) \otimes u\right) - \operatorname{div}(2\mu A u) + \nabla \pi & = 0, \\ \operatorname{div} u & = 0, \\ (\rho, u)|_{t=0} & = (\langle \rho_0 \rangle_\varepsilon, \langle u_0 \rangle_\varepsilon). \end{array} \right. \quad (2.73)$$

Moreover, thanks to the smooth transport velocity, by a similar method as in the proof of Lemma 2.2, the equation for the density ρ^ε holds in the following sense:

$$\partial_t \rho + \langle u^\varepsilon \rangle_\varepsilon \cdot \nabla \rho - \operatorname{div}(\kappa \nabla \rho) = 0 \quad \text{in } L^2(L^2).$$

In fact, we suppose a priori $\rho^\varepsilon - 1 \in L_T^\infty(H^1 \cap L^\infty) \cap L_T^2(H^2)$ for any $T \in (0, \infty)$. By use of the interpolation inequality in dimension 3

$$\|\nabla \rho\|_{L^4} \lesssim \|\Delta \rho\|_{L^2}^{1/2} \|\rho\|_{L^\infty}^{1/2}, \quad (2.74)$$

one has $\nabla \rho^\varepsilon \in L_T^4(L^4)$. Hence the scalar function $K^\varepsilon := K(\rho^\varepsilon) \in L_T^\infty(H^1 \cap L^\infty) \cap L_T^2(H^2)$ with the function K as defined in the proof of Lemma 2.2 satisfies Equation (2.68) with the transport velocity $\langle u^\varepsilon \rangle_\varepsilon$. Taking the L^2 -inner product between it and ΔK^ε , applying the following inequality (noticing $\operatorname{div}(\langle u^\varepsilon \rangle_\varepsilon) = 0$):

$$\left| \int_{\mathbb{R}^3} \langle u^\varepsilon \rangle_\varepsilon \cdot \nabla K^\varepsilon \Delta K^\varepsilon \right| = \left| \int_{\mathbb{R}^3} \nabla K^\varepsilon \cdot \nabla \langle u^\varepsilon \rangle_\varepsilon \cdot \nabla K^\varepsilon \right| \leq \|\nabla u^\varepsilon\|_{L^2} \|\nabla K^\varepsilon\|_{L^4}^2 \leq C \|\nabla u^\varepsilon\|_{L^2} \|\Delta K^\varepsilon\|_{L^2},$$

and then performing Young's Inequality and Estimate (2.57), we arrive at

$$\|\nabla K^\varepsilon\|_{L_T^\infty(L^2)} + \|\Delta K^\varepsilon\|_{L_T^2(L^2)} \leq C(\underline{\rho}, \overline{\rho}) \left(\|\nabla K^\varepsilon(0)\|_{L^2} + \|u_0\|_{L^2} \right) \leq C \left(\|\nabla \rho_0\|_{L^2} + \|u_0\|_{L^2} \right).$$

One easily finds that the above estimate also holds for ρ^ε . Therefore, taking into account also Energy Identity (1.11), we arrive at for all $T \in (0, \infty)$, $\varepsilon > 0$,

$$\|\rho^\varepsilon - 1\|_{L_T^\infty(H^1(\mathbb{R}^3))} + \|\nabla \rho^\varepsilon\|_{L_T^2(H^1(\mathbb{R}^3))} \leq C \|(\rho_0 - 1, u_0)\|_{H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)}. \quad (2.75)$$

Now we let $\varepsilon \rightarrow 0$, then the same argument as in Subsection 2.1.2 ensures that there exists a global-in-time weak solution (ρ, u) to System (1.7) such that Estimate (1.23) holds and $\nabla \rho \in C([0, \infty); L_w^2)$. Define $v = u - \kappa \nabla \ln \rho$, then one easily checks that Equality (1.21) holds for some divergence-free test function ϕ by virtue of (1.15). This completes the proof of Theorem 1.2.

⁶Due to Remark 2.3 and $u \notin L^4(L^4)$, we regularize the system in two levels in order to get the bound for the H^1 -norm of the density ρ by considering the equation for the scalar function $K = K(\rho)$ with $\nabla K = \kappa \nabla \rho$ where the transport velocity is regularized.

3 Well-posedness in dimension two

In this section we aim to prove Theorem 1.4. By the arguments before it, we just have to prove the global-in-time existence of a unique strong solution (ρ, u) to Cauchy problem (1.7)-(1.24). Indeed, by Theorem 1.3, it rests to show a pseudo-conservation law concerning $L^\infty(H^2) \times L^\infty(H^1)$ -norm of its weak solutions and moreover, such weak solutions are also strong, by use of Proposition 1.3.

The following lemma supplies the needed conservation law:

Lemma 3.1. *We assume $(\varrho, u, \nabla \pi)$ to be a weak solution to System (1.7) with the initial data ϱ_0, u_0 satisfying the following condition:*

$$0 < \underline{\varrho} \leq \varrho_0 + 1 \leq \overline{\varrho}, \quad \varrho_0 \in H^2(\mathbb{R}^2), \quad u_0 \in H^1(\mathbb{R}^2), \quad \operatorname{div} u_0 = 0, \quad (3.76)$$

then there exists one constant C_3 depending only on $\underline{\varrho}, \overline{\varrho}, \|\varrho_0\|_{H^2(\mathbb{R}^2)}, \|u_0\|_{H^1(\mathbb{R}^2)}$ such that the following a priori estimate holds true:

$$\sup_{0 \leq t \leq T} (\|\varrho\|_{H^2}^2 + \|u\|_{H^1}^2) + \int_0^T (\|\nabla \varrho\|_{H^2}^2 + \|\nabla u\|_{H^1}^2 + \|\partial_t \varrho\|_{H^1}^2 + \|\partial_t u\|_{L^2}^2 + \|\nabla \pi\|_{L^2}^2) \leq C_3. \quad (3.77)$$

Proof. It is easy to see that (2.66) already holds by Lemma 2.2. As usual, we can assume a priori that $\varrho \in L^\infty(H^2)$, $\nabla \varrho \in L^2(H^2)$ and $u \in L^\infty(H^1)$, $\nabla u \in L^2(H^1)$. In the following we will use thoroughly the Gagliardo-Nirenberg inequality (2.61) and the following interpolation inequality

$$\|f\|_{L^\infty(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}^{1/2} \|\Delta f\|_{L^2(\mathbb{R}^2)}^{1/2}. \quad (3.78)$$

We first consider the equation for u . We take the L^2 inner product between Equation (1.7)₂ and $\partial_t u$ to arrive at

$$0 = \int_{\mathbb{R}^2} \rho |\partial_t u|^2 + \frac{d}{dt} \int_{\mathbb{R}^2} 2\mu |Au|^2 + \int_{\mathbb{R}^2} \rho u \cdot \nabla u \cdot \partial_t u - \kappa \nabla \rho \cdot \nabla u \cdot \partial_t u - 2\mu' |Au|^2 \partial_t \rho. \quad (3.79)$$

On the other hand, from the equation (1.7)₂ – (1.7)₃ we have

$$\Delta u = \mu^{-1} \left(\rho \partial_t u + \rho u \cdot \nabla u - \kappa \nabla \rho \cdot \nabla u - 2\mu' \nabla \rho \cdot Au + \nabla \pi \right), \quad (3.80)$$

and the elliptic equation for π (similar to (2.49))

$$\operatorname{div}(\rho^{-1} \nabla \pi) = -\operatorname{div} \left((u - \rho^{-1} \kappa \nabla \rho) \cdot \nabla u + 2\mu \nabla \rho^{-1} \cdot Au \right). \quad (3.81)$$

Equation (3.81) above gives us the estimate for $\nabla \pi$:

$$\|\nabla \pi\|_{L^2} \leq C(\|u\|_{L^4} + \|\nabla \rho\|_{L^4}) \|\nabla u\|_{L^4}. \quad (3.82)$$

Therefore Equality (3.80) entails

$$\|\Delta u\|_{L^2} \leq C \left(\|\partial_t u\|_{L^2} + (\|u\|_{L^4} + \|\nabla \rho\|_{L^4}) \|\nabla u\|_{L^4} \right),$$

which implies, by applying (2.61) on ∇u and Young's Inequality,

$$\|\Delta u\|_{L^2} \leq C \left(\|\partial_t u\|_{L^2} + (\|u\|_{L^4}^2 + \|\nabla \rho\|_{L^4}^2) \|\nabla u\|_{L^2} \right), \quad (3.83)$$

and hence

$$\|\nabla u\|_{L^4}^2 \leq C \left(\|\partial_t u\|_{L^2} \|\nabla u\|_{L^2} + (\|u\|_{L^4}^2 + \|\nabla \rho\|_{L^4}^2) \|\nabla u\|_{L^2}^2 \right). \quad (3.84)$$

Set

$$I \triangleq \int_{\mathbb{R}^2} \rho u \cdot \nabla u \cdot \partial_t u - \kappa \nabla \rho \cdot \nabla u \cdot \partial_t u - 2\mu' |Au|^2 \partial_t \rho,$$

then

$$\|I\|_{L^1([0,T])} \leq C \int_0^T (\|u\|_{L^4} + \|\nabla \rho\|_{L^4}) \|\nabla u\|_{L^4} \|\partial_t u\|_{L^2} + \|Au\|_{L^4}^2 \|\partial_t \rho\|_{L^2}.$$

Thus by Estimate (3.84) and Young's Inequality, we finally arrive at

$$\|I\|_{L^1([0,T])} \leq \varepsilon \int_0^T \|\partial_t u\|_{L^2}^2 + C_\varepsilon \int_0^T (\|u\|_{L^4}^4 + \|\nabla \rho\|_{L^4}^4 + \|\partial_t \rho\|_{L^2}^2) \|\nabla u\|_{L^2}^2.$$

Therefore by the equality $\|\nabla u\|_{L^2}^2 = \int |Au|^2$ and Estimate (2.66), we can choose sufficiently small ε to deduce from (3.79) that

$$\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \|\partial_t u\|_{L^2}^2 \leq C e^{C_2} \|\nabla u_0\|_{L^2}^2. \quad (3.85)$$

Moreover, by estimates (3.82), (3.83), (3.84) and (3.85), we derive

$$\int_0^T \|\Delta u\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 + \|\nabla \pi\|_{L^2}^2 \leq C(\|\varrho_0\|_{H^2}, \|u_0\|_{H^1}). \quad (3.86)$$

Now we turn to the density ρ . We further apply “ Δ ” to Equation (2.68) of K , yielding the equation for the scalar function $\mathcal{K} = \Delta K \in L^\infty(L^2) \cap L^2(H^1)$:

$$\partial_t \mathcal{K} + 2\nabla u : \nabla^2 K + u \cdot \nabla \mathcal{K} + \Delta u \cdot \nabla K - \Delta(\kappa \mathcal{K}) = 0. \quad (3.87)$$

Hence again taking the L^2 inner product between (3.87) and \mathcal{K} shows

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \mathcal{K}^2 + \int_{\mathbb{R}^2} \kappa |\nabla \mathcal{K}|^2 + \int_{\mathbb{R}^2} 2(\nabla u : \nabla^2 K) \mathcal{K} + (\Delta u \cdot \nabla K) \mathcal{K} + \kappa' \mathcal{K} \nabla \rho \cdot \nabla \mathcal{K} = 0. \quad (3.88)$$

Set

$$J \triangleq \int_{\mathbb{R}^2} 2(\nabla u : \nabla^2 K) \mathcal{K} + (\Delta u \cdot \nabla K) \mathcal{K} + \kappa' \mathcal{K} \nabla \rho \cdot \nabla \mathcal{K},$$

then noticing $\|\nabla \rho\|_{L^4} \leq C \|\nabla K\|_{L^4}$, we have

$$\|J\|_{L^1([0,T])} \leq C \int_0^T (\|\nabla u\|_{L^2} \|\nabla^2 K\|_{L^4} + \|\Delta u\|_{L^2} \|\nabla K\|_{L^4} + \|\nabla K\|_{L^4} \|\nabla \mathcal{K}\|_{L^2}) \|\mathcal{K}\|_{L^4}.$$

By use of $\|\nabla^2 K\|_{L^4}, \|\mathcal{K}\|_{L^4} \lesssim \|\mathcal{K}\|_{L^2}^{1/2} \|\nabla \mathcal{K}\|_{L^2}^{1/2}$, we have from above that

$$\|J\|_{L^1([0,T])} \leq \varepsilon \int_0^T \|\nabla \mathcal{K}\|_{L^2}^2 + C_\varepsilon \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla K\|_{L^4}^4) \|\mathcal{K}\|_{L^2}^2 + \|\Delta u\|_{L^2}^2.$$

Therefore by (2.66), (2.70), (3.86) and Gronwall's Inequality, Inequality (3.88) gives

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\mathcal{K}(t)\|_{L^2}^2 + \int_0^T \|\nabla \mathcal{K}\|_{L^2}^2 &\leq C \exp \left\{ \int_0^T \|\nabla u\|_{L^2}^2 + \|\nabla K\|_{L^4}^4 \right\} \left(\|\mathcal{K}(0)\|_{L^2}^2 + \int_0^T \|\Delta u\|_{L^2}^2 \right) \\ &\leq C(\|\varrho_0\|_{H^2}, \|u_0\|_{H^1}), \end{aligned} \quad (3.89)$$

and hence

$$\sup_{0 \leq t \leq T} \|\nabla K(t)\|_{L^4} \leq \sup_{0 \leq t \leq T} \|\nabla K(t)\|_{L^2}^{1/2} \|\mathcal{K}(t)\|_{L^2}^{1/2} \leq C(\|\varrho_0\|_{H^2}, \|u_0\|_{H^1}). \quad (3.90)$$

Furthermore, by view of the two identities

$$\kappa \Delta \varrho = \mathcal{K} - \nabla \kappa \cdot \nabla \varrho \text{ and } \kappa \nabla \Delta \varrho = \nabla \mathcal{K} - \nabla \kappa \Delta \varrho - \nabla \varrho \cdot \nabla^2 \kappa - \nabla \kappa \cdot \nabla^2 \varrho,$$

we get from (2.61) and (3.78) that

$$\sup_{0 \leq t \leq T} \|\Delta \varrho(t)\|_{L^2}^2 + \int_0^T \|\nabla \Delta \varrho\|_{L^2}^2 \leq C(\|\varrho_0\|_{H^2}, \|u_0\|_{H^1}). \quad (3.91)$$

Moreover, Equation (1.7)₁ ensures

$$\nabla(\partial_t \rho) = -\nabla \rho \cdot Du - u \cdot \nabla^2 \rho + \nabla \mathcal{K},$$

which by above yields

$$\int_0^T \|\nabla \partial_t \varrho\|_{L^2}^2 \leq C(\|\varrho_0\|_{H^2}, \|u_0\|_{H^1}), \quad (3.92)$$

which together with the estimates (2.66), (3.85), (3.86) and (3.91) give (3.77). \square

Remark 3.1. *It is easy to deduce from (3.81) that*

$$-\rho^{-1} \Delta \pi = \nabla \rho^{-1} \cdot \nabla \pi + (\nabla u - \kappa \rho^{-1} \nabla \rho) : \nabla u - \mu \nabla \rho^{-1} \cdot \Delta u.$$

Thus by the uniform bound (3.77), one gets $\|\Delta \pi\|_{L^1(L^2)} \leq C(\|\varrho_0\|_{H^2(\mathbb{R}^2)}, \|u_0\|_{H^1(\mathbb{R}^2)})$.

The next lemma is devoted to the weak-strong uniqueness result under the initial condition (3.76).

Lemma 3.2. *Under the same hypotheses of Lemma 3.1, the weak solutions belong to (strong) solution space E_T for any $T \in (0, +\infty)$.*

Proof. Denote by C_0 some harmless constant depending only on $\|\varrho_0\|_{H^2(\mathbb{R}^2)}, \|u_0\|_{H^1(\mathbb{R}^2)}$ in the following. Let's rewrite the equations for ρ and u in System (1.7) as following:

$$\begin{cases} \partial_t \rho - \operatorname{div}(\kappa \nabla \rho) = -u \cdot \nabla \rho, \\ \partial_t u - \operatorname{div}(\mu \rho^{-1} \nabla u) = -(u - (\kappa \rho^{-1} + \mu \rho^{-2}) \nabla \rho) \cdot \nabla u - \rho^{-1} \nabla \mu \cdot Du - \rho^{-1} \nabla \pi. \end{cases} \quad (3.93)$$

By Proposition 1.1 and Proposition 1.4, the $L_T^1(B_{2,1}^1)$ (resp. $L_T^1(B_{2,1}^0)$)-norm of the right hand side of Equation (3.93)₁ (resp. (3.93)) can be bounded by the following two quantities respectively:

$$C \int_0^T \|u\|_{B_{2,1}^1} \|\nabla \rho\|_{B_{2,1}^1} dt \quad \text{and} \quad C \int_0^T \left((\|u\|_{B_{2,1}^1} + \|\varrho\|_{B_{2,1}^2}) \|\nabla u\|_{B_{2,1}^0} + \|\varrho\|_{B_{2,1}^1} \|\nabla \pi\|_{B_{2,1}^0} \right) dt.$$

Since $\|\varrho\|_{B_{2,1}^1} \leq \|\varrho\|_{H^2}$ and $\|\nabla \pi\|_{B_{2,1}^0} \lesssim \|\nabla \pi\|_{L^2} + \|\Delta \pi\|_{L^2}$, we have from Estimate (3.77) and Equation (3.81) that

$$\int_0^T \|\varrho\|_{B_{2,1}^1} \|\nabla \pi\|_{B_{2,1}^0} dt \lesssim \int_0^T C_0 \left(\|\Delta \pi\|_{L^2} + \|(u - \kappa \rho^{-1} \nabla \rho) \cdot \nabla u + 2\mu \nabla \rho^{-1} \cdot Au\|_{L^2} \right) dt.$$

Because $B_{2,1}^0 \hookrightarrow L^2$, the product estimates entails

$$\int_0^T \|\varrho\|_{B_{2,1}^1} \|\nabla \pi\|_{B_{2,1}^0} dt \leq C_0 \int_0^T \|\Delta \pi\|_{L^2} + C_0 \int_0^T (\|u\|_{B_{2,1}^1} + \|\varrho\|_{B_{2,1}^2}) \|\nabla u\|_{B_{2,1}^0} dt.$$

On the other side, it is easy to check that for any $C^1(\mathbb{R}, \mathbb{R})$ -function f with $f(1) = 0$,

$$\|f(\rho)\|_{L^\infty(H^2)} + \|\nabla f(\rho)\|_{L^2(H^2)} \leq C_0.$$

Therefore, for any solution (ϱ, u) to System (1.7), Proposition 1.3 tells us that

$$\begin{aligned} & \|\varrho\|_{L_T^\infty(B_{2,1}^1) \cap L_T^1(B_{2,1}^3)} + \|u\|_{L_T^\infty(B_{2,1}^0) \cap L_T^1(B_{2,1}^2)} \\ & \leq C_0 \left(1 + \int_0^T \|(\varrho, u)\|_{L^2} + \int_0^T \|u\|_{B_{2,1}^1} \|\nabla \rho\|_{B_{2,1}^1} + (\|u\|_{B_{2,1}^1} + \|\varrho\|_{B_{2,1}^1}) \|\nabla u\|_{B_{2,1}^0} + \|\Delta \pi\|_{L^2} \right). \end{aligned}$$

As we have interpolation inequalities

$$\|u\|_{B_{2,1}^1} \lesssim \|u\|_{B_{2,1}^0}^{1/2} \|u\|_{B_{2,1}^2}^{1/2}, \quad \|\varrho\|_{B_{2,1}^2} \lesssim \|\varrho\|_{B_{2,1}^1}^{1/2} \|\varrho\|_{B_{2,1}^3}^{1/2},$$

hence, by Young's Inequality and Gronwall's Inequality again, the above inequality implies

$$\begin{aligned} & \|\varrho\|_{L_T^\infty(B_{2,1}^1) \cap L_T^1(B_{2,1}^3)} + \|u\|_{L_T^\infty(B_{2,1}^0) \cap L_T^1(B_{2,1}^2)} \\ & \leq C_0 \exp \left\{ \int_0^T \|\nabla \rho\|_{B_{2,1}^1}^2 + \|\nabla u\|_{B_{2,1}^0}^2 \right\} \left(1 + \|(\varrho, u)\|_{L_T^1(L^2)} + \|\Delta \pi\|_{L_T^1(L^2)} \right). \end{aligned}$$

It is easy to find from Estimate (3.77) and Remark 3.1 that

$$\|\varrho\|_{L_T^\infty(B_{2,1}^1) \cap L_T^1(B_{2,1}^3)} + \|u\|_{L_T^\infty(B_{2,1}^0) \cap L_T^1(B_{2,1}^2)} \leq C_0 \left(1 + \|(\varrho, u)\|_{L_T^1(L^2)} \right).$$

This completes the proof. \square

Now with Lemma 3.2 in hand, we will check that there exists a globally existing strong solution to Cauchy problem (1.7)-(1.24). Firstly, Theorem 1.3 ensures that there exists a unique strong solution $(\rho_1, u_1, \nabla \pi_1)$ on its lifespan $[0, T^*)$ for some positive time $T^* > T_c$. Hence there exists $T_0 \in (0, T^*)$ such that $\varrho_1(T_0) \in B_{2,1}^2 \hookrightarrow H^2$ and $u_1(T_0) \in B_{2,1}^1 \hookrightarrow H^1$. Let $(\rho_2, u_2, \nabla \pi_2)$ to be a weak solution which evolves from the initial data $\rho_1(T_0), u_1(T_0)$. Thus, since Lemma 3.2 ensures that $(\rho_2, u_2, \nabla \pi_2) \in E_T$ for any positive finite T , it coincides with the strong solution $(\rho_1, u_1, \nabla \pi_1)$ on the time interval $[T_0, T^*)$ by the uniqueness result. Furthermore, we can choose $T^* = +\infty$. In fact, if $T^* < +\infty$, then by the global-in-time boundedness of $\|\rho_2 - 1\|_{L^\infty(B_{2,1}^1)}$ and $\|u_2\|_{L^\infty(B_{2,1}^0)}$ given by (3.77), the lifespan of $(\varrho_1, u_1, \nabla \pi_1)$ should go beyond $[0, T^*)$. This is a contradiction.

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